1988

Theory and analysis of consecutive-k-out-of-n:G systems reliability

Weixing Zhang
Iowa State University

Order Number 8825470

Theory and analysis of consecutive-k-out-of-n:G systems reliability

Zhang, Weixing, Ph.D.

Iowa State University, 1988

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

Theory and analysis of consecutive-k-out-of-n:G systems reliability

by

Weixing Zhang

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major: Industrial Engineering

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1988

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

I. INTRODUCTION

Reliability problems become more and more important, especially in complex and high technology systems. They are particularly critical when there are concerns over the consequences of system failures in terms of safety and cost. The tragedy of space shuttle Challenger is the best example.

There are many types of systems in reliability evaluation: series systems, parallel systems, series-parallel systems, parallel-series systems, complex systems, and some special systems such as k-out-of-n systems and consecutive-k-out-of-n:F systems. Evaluation of system reliability depends on the relationship between the components of a system and their effect on the system, i.e., system structure and component reliabilities.

Recently, there has been a considerable interest in the consecutive-k-out-of-n:F systems. Although there was some research on other topics related to the consecutive-k-out-of-n:F systems before 1980, the first article devoted to this field should be attributed to Kontoleon [12]. Since 1980, a lot of research has been conducted by many people [1,3-6,8-11,13-19,21] and they have been concentrating on reliability evaluation of the system and bounds on system reliability. A few authors discussed some aspects of the optimal sequencing of the system [6,14,15,21].

A consecutive-k-out-of-n:F system is a sequence of n ordered components such that the system works if and only if less than k consecutive components fail. One application of such systems is an oil pipeline system with n pump stations. Each station is powerful enough to send oil as far as to the next k pump stations. If less than k consecutive stations fail, the flow of oil will not be interrupted and the pipeline system will still function properly. The configuration of a linear consecutive-2-out-of-8:F system is given in Figure 1.1.

FIGURE 1.1. A linear consecutive-2-out-of-8:F system

This research introduces a special system: consecutive-k-out-of-n:G system and develops the basic theory of the consecutive-k-out-of-n:G systems reliability. A consecutive-k-out-of-n:G system consists of an ordered sequence of n components such that the system works whenever at least k consecutive components in the system are good. The system can be either a linear system or a circular system, depending on whether all components are linearly arranged or circularly arranged.

There exist applications of the consecutive-k-out-of-n:G systems. One example is a railway station of n lines. Because of particular requirements, a special train can enter the station only if at least k lines are available (or empty); otherwise, the station fails to receive the train.

The objectives of this study are as follows.

1. Introduce the concept of consecutive-k-out-of-n:G systems.
2. Derive the methods to evaluate the reliability of the consecutive-k-out-of-n:G systems.
3. Study properties of consecutive-k-out-of-n:G systems.
4. Investigate relationship between the consecutive-k-out-of-n:G systems and the consecutive-k-out-of-n:F systems.
5. Derive the principles for the optimal system design.

II. EVALUATION OF RELIABILITY OF THE LINEAR CONSECUTIVE-k-OUT-OF-n:G SYSTEM

A. Introduction

This study introduces a new special system: the consecutive-k-out-of-n:G system which functions whenever at least k consecutive components are in operation.

A street parking system as shown in Figure 2.1 is a good example of such systems. Suppose that there are seven parking spaces on a street. Each space is suitable for one car. If a bus parks on the street, it will take two spaces. Every parking space has a probability that it is not occupied. An interesting problem is to find the probability that the bus can park on this street.

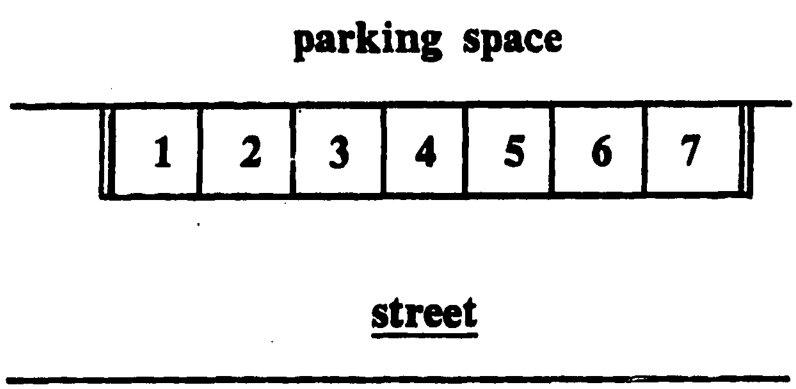

FIGURE 2.1. Street parking system

In fact, this is the reliability problem of a linear consecutive-2-out-of-7:G system. The bus can park if and only if at least two consecutive parking spaces on the street are empty. The configuration of this linear consecutive-2-out-of-7:G system is shown in Figure 2.2.

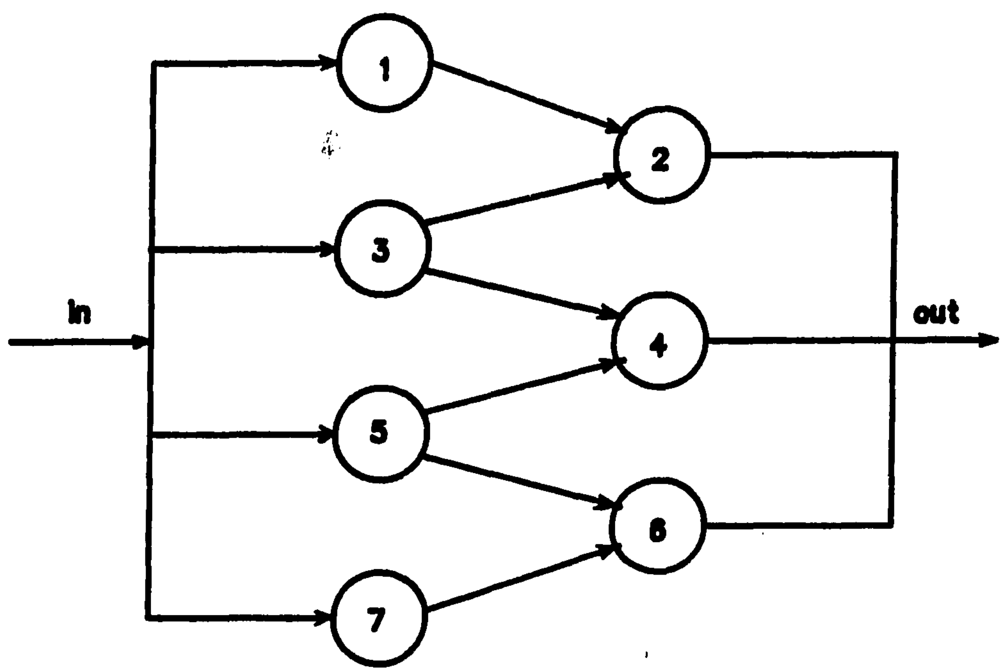

FIGURE 2.2. A linear consecutive-2-out-of-7:G system

For a system, consisting of n components, each component has two states (operation or failing). Thus, there are 2^n possible states for the system. Calculations of the system reliability require consideration of all states in which the system functions. There are many methods to evaluate the reliability of a system and some are more efficient than the others. For the special system proposed here, the efficient methods will be derived.

B. Notation and Assumptions

n number of components in a system

k minimum number of consecutive good components required for the system to function

r_i reliability of component i in the system, $i=1,2,\ldots,n$

u_i unreliability of component i; $u_i=1-r_i$

$R(j;k)$ reliability of a consecutive-k-out-of-j:G system where $j=0,1,2,\ldots,n$

$R(r_1,\ldots,r_j;k)$ same as $R(j;k)$, with component reliabilities explicitly expressed by $r_1,r_2,\ldots,r_j$.

$Q(j;k)$ unreliability of the system; $Q(j;k)=1-R(j;k)$

$Q(r_1,\ldots,r_j;k)$ same as $Q(j;k)$, with component reliabilities explicitly expressed by $r_1,r_2,\ldots,r_j$.

X_i state of component i:

$$= \begin{cases} 0, & \text{if component i fails} \\ 1, & \text{if component i is good} \end{cases}$$

T random variable which represents the position of the last functioning component in the sequence of n components; $T=t$, $t=0,1,2,\ldots,n$.

M random variable which represents the position of the last failed component in the sequence of t-1 components before component t; M=m, m=0,1,2,...,t-1.

It is assumed that:

- There are only two states, operational or failing, for a component or a system.
- X_1, X_2, ..., X_n are mutually independently, but not necessarily identically distributed, i.e., u_i's may be different.

C. Computation of Reliability

Consider a system with n linearly arranged components. The components are numbered from 1 to n. Component i works with probability r_i and fails with probability u_i. The system operates whenever there are at least k consecutive good components in the system.

The system reliability is given by Theorem 2.1, followed by a proof. In addition, another approach to the system reliability is presented in this section.

Theorem 2.1:

For a linear consecutive-k-out-of-n:G system where all components are not necessarily identical, the reliability of the system is given as follows:

$$R(n;k)=R(n-1;k)+Q(n-k-1;k)u_{n-k}\left(\prod_{i=n-k+1}^{n} r_i\right) \tag{2.1}$$

or

$$R(n;k)=R(r_2,\ldots,r_n;k)+(\prod_{i=1}^{k} r_i)u_{k+1}Q(r_{k+2},\ldots,r_n;k)$$

(2.2)

Corollary 1:

If all components are equally reliable in the system, i.e., $r_1=r_2=\ldots=r_n=r$, then

$$R(n;k)=R(n-1;k)+ur^k[1-R(n-k-1;k)]$$

(2.3)

Proof of Theorem 2.1:

Using the factorization probability theorem [20],

$$R(r_1,\ldots,r_n;k)=u_nR(r_1,\ldots,r_{n-1},0;k)+r_nR(r_1,\ldots,r_{n-1},1;k)$$

(2.4)

By the definition of a linear consecutive-k-out-of-n:G system,

$$R(r_1,\ldots,r_{n-1},0;k)=R(r_1,\ldots r_{n-1};k)=R(n-1;k)$$

(2.5)

$$R(r_1,\ldots,r_{n-1},1;k)=R(r_1,\ldots,r_{n-1};k)+Pr\{E_1 \text{ and } E_2 \text{ and } E_3\}$$

where E_1 is the event that exactly k-1 consecutive good components are adjacent to component n, E_2 the event that component n-k is failed, and E_3 the event that the remaining n-k-1 components do not constitute any sequence of at least k consecutive good components.

The event of a functioning linear consecutive-k-out-of-(n-1):G system consisting of component 1 through n-1, and the event of $\{E_1$ and E_2 and $E_3\}$ are disjoint. Therefore,

$R(r_1,\ldots,r_{n-1},1;k)$

$$=R(r_1,\ldots,r_{n-1};k)+(\prod_{i=n-k+1}^{n-1} r_i)u_{n-k}Q(r_1,\ldots r_{n-k-1};k) \tag{2.6}$$

Substitute equations (2.5) and (2.6) into equation (2.4) and we obtain

$$R(n;k)=u_nR(n-1;k)$$

$$+r_n[R(r_1,\ldots,r_{n-1};k)+(\prod_{i=n-k+1}^{n-1} r_i)u_{n-k}Q(r_1,\ldots,r_{n-k-1};k)]$$

$$=R(n-1;k)+Q(n-k-1;k)u_{n-k}(\prod_{i=n-k+1}^{n} r_i)$$

In a similar way, equation (2.2) can be proved.

Q.E.D.

Proof of Corollary 1 automatically follows if all components in the system have the same life distribution.

The derivation described below produces the same result.

A scheme to facilitate the reliability calculation is given in Figure 2.3.

There are n components in sequence where 0's indicate failed states of the components and 1's good states of the components. In terms of unreliability,

$$Q(n;k) = Pr(\text{system fails})$$

$$= \sum_t \sum_m Pr(\text{system fails}/T=t,M=m)\cdot Pr(T=t,M=m)$$

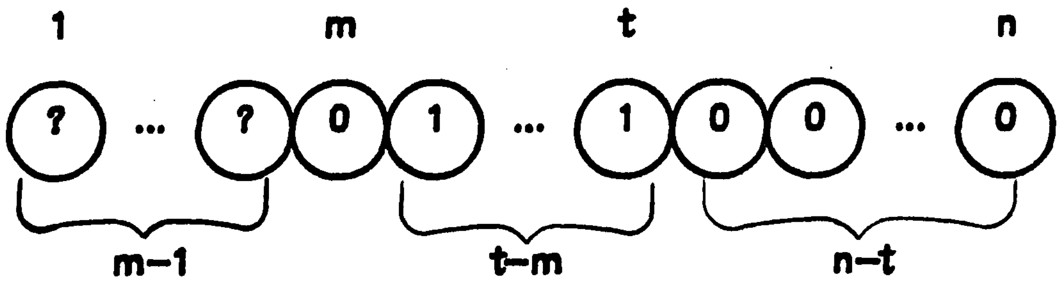

FIGURE 2.3. Scheme of linear system state.

$$= \sum_t \sum_m Q(n;k/T=t,M=m) \cdot Pr(T=t,M=m)$$

(2.7)

First, we derive unreliabilities of the system based on the mutually exclusive events on the conditions of M and T over all values of m and t. Then, those conditional unreliabilities are used to construct the unreliability of the system. The conditional unreliability of the system, Pr(system fails/T=t,M=m), will be zero if $t-m \geq k$, i.e., $t \geq m+k$, since at least k consecutive good components will guarantee that the system functions. If $t<k$, it is clear that the system has less than k consecutive good components so that Pr(system fails/T=t,M=m)=1 for $t<k$. Further, for the situation of $k \leq t \leq m+k-1$, the event that the whole system fails is equivalent to the event that the subsystem of the first m-1 components in sequence fails. Therefore, this leads to:

$$Q(n;k/T=t,M=m) = Q(m-1;k) \quad \text{for} \quad k \leq t \leq m+k-1$$

In summary,

$$Q(n;k/T=t,M=m) = \begin{cases} 0, & t\geq m+k, \text{ i.e., } m\leq t-k \\ 1, & t<k \\ Q(m-1;k) & k\leq t\leq m+k-1 \end{cases}$$

Using the above results, one can get the following:

$$Q(n;k) = \sum_t \sum_m Q(n;k/T=t,M=m)\cdot Pr(T=t,M=m)$$

$$= \sum_{t<k} \sum_m Q(n;k/T=t,M=m)\cdot Pr(T=t,M=m)$$

$$+ \sum_{t=k}^{n} \sum_{m=0}^{t-k} Q(n;k/T=t,M=m)\cdot Pr(T=t,M=m)$$

$$+ \sum_{t=k}^{n} \sum_{m=t-k+1}^{t-1} Q(n;k/T=t,M=m)\cdot Pr(T=t,M=m)$$

$$= \sum_{t<k} \sum_m Q(n;k/T=t,M=m)\cdot Pr(T=t,M=m)$$

$$+ \sum_{t=k}^{n} \sum_{m=t-k+1}^{t-1} Q(n;k/T=t,M=m)\cdot Pr(T=t,M=m)$$

$$= \prod_{i=k}^{n} u_i + \sum_{t=k}^{n} \sum_{m=t-k+1}^{t-1} Q(n;k/T=t,M=m)\cdot Pr(T=t,M=m)$$

$$= \sum_{t=k}^{n} \sum_{m=t-k+1}^{t-1} Q(m-1;k)u_m\left(\prod_{i=t+1}^{n} u_i\right)\left(\prod_{i=m+1}^{t} r_i\right) + \prod_{i=k}^{n} u_i$$

$$= \sum_{t=k}^{n-1} \sum_{m=t-k+1}^{t-1} Q(m-1;k)u_m\left(\prod_{i=t+1}^{n} u_i\right)\left(\prod_{i=m+1}^{t} r_i\right) + \prod_{i=k}^{n} u_i$$

$$+ \sum_{m=n-k+1}^{n-1} Q(m-1;k)u_m(\prod_{i=m+1}^{n} r_i)$$

(when t>n, the sequence t+1, t+2, ..., t+n is empty)

$$= \sum_{t=k}^{n-1} \sum_{m=t-k+1}^{t-1} Q(m-1;k)u_m(\prod_{i=t+1}^{n-1} u_i)(\prod_{i=m+1}^{t} r_i)u_n + (\prod_{i=k}^{n-1} u_i)u_n$$

$$+ \sum_{m=n-k+1}^{n-1} Q(m-1;k)u_m(\prod_{i=m+1}^{n} r_i)$$

$$= Q(n-1;k)u_n + \sum_{m=n-k+1}^{n-1} Q(m-1;k)u_m(\prod_{i=m+1}^{n} r_i)$$

$$= \sum_{m=n-k+1}^{n} Q(m-1;k)u_m(\prod_{i=m+1}^{n} r_i)$$

$$= Q(n-1;k)u_n$$

$$+[\sum_{m=(n-1)-k+1}^{n-1} Q(m-1;k)u_m(\prod_{i=m+1}^{n-1} r_i)]r_n$$

$$- Q(n-k-1;k)u_{n-k}(\prod_{i=n-k+1}^{n} r_i)$$

$$= Q(n-1;k)r_n + Q(n-1;k)u_n - Q(n-k-1;k)u_{n-k}(\prod_{i=n-k+1}^{n} r_i)$$

$$= Q(n-1;k) - Q(n-k-1;k)u_{n-k}\left(\prod_{i=n-k+1}^{n} r_i\right)$$

(2.8)

Then, the reliability of the consecutive-k-out-of-n:G system is:

$$R(n;k)=1-Q(n;k)$$

$$=1-Q(n;k)+Q(n-k-1;k)u_{n-k}\left(\prod_{i=n-k+1}^{n} r_i\right)$$

$$=R(n-1;k)+Q(n-k-1;k)u_{n-k}\left(\prod_{i=n-k+1}^{n} r_i\right)$$

(2.9)

If all components in the system have the same reliability, r, then we have:

$$R(n;k)=R(n-1;k)+Q(n-k-1;k)ur^k$$

(2.10)

For consecutive-k-out-of-n:G systems, there are some special cases. If k=1, then the system is in fact a parallel system and if k=n, it is a series system. When n<k, the reliability of the system, R(n;k), is zero.

D. Example

A consecutive-k-out-of-n:G system, with k=3 and n=5, is considered here. All components have the same reliability, r.

$$R(0;3)=R(1;3)=R(2;3)=0$$

$$R(3;3)=r^3$$

$$R(4;3)=R(3;3)+Q(0;3)(1-r)r^3$$

$$=R(3;3)+[1-R(0;3)](1-r)r^3$$

$$=r^3+(1-r)r^3$$

$$=2r^3-r^4$$

$$R(5;3)=R(4;3)+[1-R(1;3)](1-r)r^3$$
$$=2r^3-r^4+(1-r)r^3$$
$$=2r^3-r^4+r^3-r^4$$
$$=3r^3-2r^4$$

Another approach to get the solution is to first list all $2^5=32$ possible system states, and then sort out those states in which the system works. In this particular example, 8 out of 32 possible states contribute to the system reliability and are listed as follows:

1 1 1 0 0	$r^3(1-r)^2$
0 1 1 1 0	$r^3(1-r)^2$
1 1 1 1 0	$r^4(1-r)$
1 1 1 0 1	$r^4(1-r)$
0 0 1 1 1	$r^3(1-r)^2$
1 0 1 1 1	$r^4(1-r)$
0 1 1 1 1	$r^4(1-r)$
1 1 1 1 1	r^5

The system reliability is the summation of those mutually exclusive state probabilities, i.e.,

$$R(5;3)=3r^3(1-r)^2+4r^4(1-r)+r^5$$
$$=3r^3-2r^4,$$

which is the same as that obtained by the method introduced in Theorem 2.1. As n becomes larger, the computational efficiency of this introduced method becomes very appealing.

E. Approximation in the i.i.d. Case

1. Approximation in large linear systems

When the number of components in the system is large and all components have the same failure distribution, i.e., $r_1=r_2=\ldots=r_n=r$, a result from Feller [FE1] can be modified to approximate the system reliability of the consecutive-k-out-of-n:G system. In this case, the system reliability is:

$$R(n;k) \cong 1-(1-rx)/[(k+1-kx)(1-r)x^{n+1}]$$

$$= \frac{(k+1-kx)(1-r)x^{n+1}-1+rx}{(k+1-kx)(1-r)x^{n+1}} \tag{2.11}$$

where x is a positive solution to the equation

$$1-(1-r)x\sum_{i=0}^{k-1} r^i x^i=0$$

Using the summation equation for a finite geometric series, the above equation can be written as:

$$1-(1-r)x\left(\frac{1-(rx)^k}{1-rx}\right) = 0$$

$$\text{i.e., } (1-r)r^k x^{k+1}-x+1=0 \tag{2.12}$$

Now, the problem becomes solving equation (2.12) first for a positive solution and then using the solution to obtain the answer from equation (2.11). By observation, $x=1/r$ is a positive root of equation (2.12). However, if $x=1/r$, then $R(n;k)=1$. This is not what we want.

The unique positive solution different from $x=1/r$ in equation (2.12) must be sought to approximate the system reliability.

There are a number of techniques which can be used to solve equation (2.12). Here, the Newton's Method is employed.

$$x_{j+1}=x_j-f(x_j)/f'(x_j), \qquad j=0,1,2,\ldots \tag{2.13}$$

In this particular situation,

$$x_{j+1}=x_j-\frac{(1-r)r^k x_j{}^{i+1} x_j+1}{(k+1)(1-r)r^k x_j{}^{k-1}} \qquad j=0,1,2,\ldots \tag{2.14}$$

Table 2.1 gives the exact reliability (upper entry) and the approximated reliability (lower entry) for given values of n and k with a known i.i.d. component reliability, $r=0.5$. Table 2.2 is the results for $r=0.65$. From these tables, we can see that the system reliabilities are approximated quite well.

2. Error analysis in approximation

Equations (2.11) and (2.12) were derived from the model of success runs in Bernoulli trials [7]. In the model, E stands for the occurrence of a success run of length k in a sequence of Bernoulli trials. The f_n is defined as the probability that the first run of length k occurs at the n^{th} trial, i.e.,

$$f_n = Pr\{\text{ E occurs for the first time at the } n^{th} \text{ trial }\}$$

In fact, n, the number of trials, is a random variable and when $k=1$ it follows the geometric distribution. It can not be determined in which trial E occurs, since it is possible that n is infinite.

TABLE 2.1. Comparison 1 of exact and approximated reliabilities

COMPONENT RELIABILITY = 0.500	k=2	k=3	k=4	k=5	k=6
n= 2	0.250000				
	0.233688				
n= 3	0.375000	0.125000			
	0.380041	0.115308			
n= 4	0.500000	0.187500	0.062500		
	0.498442	0.186398	0.058105		
n= 5	0.593750	0.250000	0.093750	0.031250	
	0.594231	0.251776	0.092218	0.029477	
n=10	0.859375	0.507813	0.245117	0.109375	0.046875
	0.859373	0.507812	0.245124	0.109329	0.046898
n=15	0.951263	0.676239	0.372284	0.182617	0.085388
	0.951263	0.676235	0.372274	0.182610	0.085374
n=20	0.983109	0.787028	0.478019	0.249870	0.122315
	0.983109	0.787025	0.478008	0.249863	0.122296

TABLE 2.2. Comparison 2 of exact and approximated reliabilities

COMPONENT RELIABILITY = 0.650	k=2	k=3	k=4	k=5	k=6
n= 2	0.422500				
	0.387916				
n= 3	0.570375	0.274625			
	0.581909	0.243967			
n= 4	0.718250	0.370744	0.178506		
	0.714419	0.368899	0.157450		
n= 5	0.803648	0.466862	0.240983	0.116029	
	0.804930	0.473186	0.234841	0.102916	
n=10	0.970999	0.786437	0.527161	0.319080	0.181005
	0.970994	0.786475	0.527359	0.318343	0.181763
n=15	0.995687	0.913457	0.708050	0.482079	0.304925
	0.995687	0.913456	0.708048	0.482037	0.304940
n=20	0.999358	0.964923	0.819663	0.606431	0.409587
	0.999359	0.964922	0.819660	0.606421	0.409575

An exact expression for probability f_n involves k terms and usually the calculations of all terms are prohibitive. Fortunately, a single term almost always provides a satisfactory approximation [7]. Therefore,

$$f_n \cong \frac{(x-1)(1-rx)}{(k+1-kx)(1-r)} \cdot \frac{1}{x^{n+1}} \tag{2.15}$$

The probability of no run in n trials is

$$Q(n;k) = f_{n+1} + f_{n+2} + f_{n+3} + \ldots$$

$$\cong \frac{1-rx}{(k+1-kx)(1-r)} \cdot \frac{1}{x^{n+1}} \tag{2.16}$$

where x is the smallest root of equation (2.12).

From the model of success runs in Bernoulli trials, it is clear that the approximation improves with n. The larger the n value is, the better the approximation is.

Even for a very very large n, neglecting k-1 terms in equation (2.15) contributes some error in approximation to f_n. Feller showed that in equation (2.16), the error committed by neglecting k-1 terms in equation (2.15) is less in absolute value than

$$\frac{2(k-1)r}{k(1-r)(1+r)}$$

In fact, from this upper bound of error in approximation, we can restrict the error in approximating the system reliability.

Let

$$f(r)= \frac{2(k-1)r}{k(1-r^2)} \tag{2.17}$$

$$f'(r) = \frac{2(k-1)(1+r)^2}{k(1-r^2)^2} \tag{2.18}$$

$$f''(r) = \frac{4(k-1)r(3+r^2)}{k(1-r^2)^3} \tag{2.19}$$

Since both $f'(r)$ and $f''(r)$ are non-negative, $f(r)$ is convex. This implies that as component reliability increases, the upper bound increases even faster. Actually, $f(r)$ can be so large as to have no practical meaning.

For a given value of k, it is desired to determine an upper bound for component reliability r such that

$$f(r) \le c, \quad 0 < c \le 1 \tag{2.20}$$

$$\frac{2(k-1)r}{k(1-r^2)} \le c$$

$$\frac{2(k-1)r}{ck} \le 1-r^2$$

$$r^2 + \frac{2(k-1)r}{ck} - 1 \leq 0. \tag{2.21}$$

Finally, we find that

$$r \leq \frac{1-k}{ck} + \sqrt{1+\left(\frac{k-1}{ck}\right)^2} \tag{2.22}$$

It appears that when component reliability is small, the upper bound of error in approximation is small. This is also true for small k value. However, after k reaches a certain value, for a fixed component reliability value, the upper bound of error increases only slightly.

It is recommended from this study that for a very large system with a fixed k value, inequality (2.22) be used to check whether or not the approximation to system reliability is satisfactory. If not, the exact system reliability can be obtained by using recursive methods.

F. Closed Formulas for Computing Reliability in the i.i.d. Case

When using formula (2.3) to compute the reliability of a linear consecutive-k-out-of-n:G system with equally reliable components, the reliabilities of the systems with n-1, n-2, ..., k equally reliable components must be computed first so that the reliability of n component system can be obtained. However, we are not interested in the reliabilities with n-1, n-2, ..., k component systems. Formula (2.3) is further analyzed and some results are derived.

By formula (2.3),

$$R(n;k)=R(n-1;k)+[1-R(n-k-1;k)]ur^k$$

$$=R(n-1;k)+Q(n-k-1;k)ur^k$$

Since $Q(n;k)=1$ when $n<k$, then we have:

$$R(n;k)=R(n-1;k)+ur^k \quad \text{for } k<n\leq 2k$$

$$=R(n-2;k)+ur^k+ur^k$$

$$=r^k+(n-k)ur^k$$

Since $Q(n;k)=1-r^k-(n-k)ur^k$ for $k\leq n\leq 2k$, we have

$$R(n;k)=R(n-1;k)+ur^kQ(n-k-1;k)$$

$$=r^k+kur^k[\sum_{i=0}^{n-2k-1}(1-r^k-iur^k)]$$

In summary,

$$R(n;k)=0, \quad \text{if } n<k \tag{2.23a}$$

$$R(n;k)=r^k, \quad \text{if } n=k \tag{2.23b}$$

$$R(n;k)=r^k+(n-k)ur^k, \quad \text{if } k\leq n\leq 2k \tag{2.23c}$$

$$R(n;k)=r^k+kur^k[\sum_{i=0}^{n-2k-1}(1-r^k-iur^k)],$$

$$\text{if } 2k+1 \leq n \leq 3k \tag{2.23d}$$

$$R(n;k)=R(n-1;k)+ur^k[1-R(n-k-1;k)], \quad \text{if } n>3k \tag{2.23e}$$

Therefore, for a linear consecutive-k-out-of-n:G system, if it satisfies condition: $n<3k+1$, then the system reliability can be computed in a closed formula.

Reliabilities of linear consecutive-k-out-of-n:G systems with equal component reliability r are tabulated in Table 2.3 ($r=0.65$), Table 2.4 ($r=0.8$), Table 2.5 ($r=0.95$), and Table 2.6 ($r=0.99$).

TABLE 2.3. Reliabilities of linear consecutive-k-out-of-n:G systems (r=0.65)

COMPONENT RELIABILITY = 0.650	k=2	k=3	k=4	k=5	k=6	k=7
n= 2	0.422500					
n= 3	0.570375	0.274625				
n= 4	0.718250	0.370744	0.178506			
n= 5	0.803648	0.466862	0.240983	0.116029		
n= 6	0.867178	0.562981	0.303461	0.156639	0.075419	
n= 7	0.908842	0.632703	0.365938	0.197249	0.101815	0.049022
n= 8	0.937878	0.693186	0.428415	0.237859	0.128212	0.066180
n= 9	0.957519	0.744431	0.479739	0.278470	0.154609	0.083338
n=10	0.970999	0.786437	0.527161	0.319080	0.181005	0.100495
n=11	0.980185	0.821741	0.570678	0.354978	0.207402	0.117653
n=12	0.986467	0.851231	0.610293	0.389227	0.233798	0.134811
n=13	0.990755	0.875796	0.646004	0.421827	0.258204	0.151969
n=14	0.993686	0.896323	0.678508	0.452777	0.281913	0.169126
n=15	0.995687	0.913457	0.708050	0.482079	0.304925	0.185443
n=16	0.997054	0.927757	0.734872	0.509731	0.327241	0.201465
n=17	0.997987	0.939695	0.759220	0.535925	0.348859	0.217193
n=18	0.998625	0.949660	0.781337	0.560729	0.369781	0.232627
n=19	0.999061	0.957979	0.801423	0.584208	0.390006	0.247766
n=20	0.999358	0.964923	0.819663	0.606431	0.409587	0.262610

TABLE 2.4. Reliabilities of linear consecutive-k-out-of-n:G systems (r=0.80)

COMPONENT RELIABILITY = 0.800

	k=2	k=3	k=4	k=5	k=6	k=7
n= 2	0.640000					
n= 3	0.768000	0.512000				
n= 4	0.896000	0.614400	0.409600			
n= 5	0.942080	0.716800	0.491520	0.327680		
n= 6	0.971776	0.819200	0.573440	0.393216	0.262144	
n= 7	0.985088	0.869171	0.655360	0.458752	0.314573	0.209715
n= 8	0.992501	0.908656	0.737280	0.524288	0.367001	0.251658
n= 9	0.996114	0.937656	0.785645	0.589824	0.419430	0.293601
n=10	0.998023	0.956170	0.827300	0.655360	0.471859	0.335544
n=11	0.998983	0.969567	0.862244	0.699421	0.524287	0.377487
n=12	0.999480	0.978920	0.890476	0.739187	0.576716	0.419430
n=13	0.999733	0.985304	0.911998	0.774658	0.615401	0.461373
n=14	0.999863	0.989792	0.929558	0.805834	0.651337	0.503316
n=15	0.999930	0.992909	0.943706	0.832716	0.684524	0.536463
n=16	0.999964	0.995067	0.954991	0.855302	0.714963	0.567850
n=17	0.999981	0.996572	0.963963	0.875001	0.742653	0.597479
n=18	0.999990	0.997617	0.971172	0.892093	0.767594	0.625348
n=19	0.999995	0.998343	0.976943	0.906861	0.789786	0.651458
n=20	0.999997	0.998849	0.981554	0.919586	0.809950	0.675809

TABLE 2.5. Reliabilities of linear consecutive-k-out-of-n:G systems (r=0.95)

COMPONENT RELIABILITY = 0.950	k=2	k=3	k=4	k=5	k=6	k=7
n= 2	0.902500					
n= 3	0.947625	0.857375				
n= 4	0.992750	0.900244	0.814506			
n= 5	0.997150	0.943112	0.855231	0.773781		
n= 6	0.999513	0.985981	0.895957	0.812470	0.735091	
n= 7	0.999840	0.992095	0.936682	0.851159	0.771846	0.698337
n= 8	0.999969	0.996372	0.977407	0.889848	0.808601	0.733254
n= 9	0.999991	0.998810	0.984962	0.928537	0.845355	0.768171
n=10	0.999998	0.999411	0.990857	0.967226	0.882110	0.803087
n=11	0.999999	0.999750	0.995095	0.975978	0.918864	0.838004
n=12	1.000000	0.999906	0.997673	0.983233	0.955619	0.872921
n=13	1.000000	0.999957	0.998593	0.988992	0.965356	0.907838
n=14	1.000000	0.999982	0.999206	0.993254	0.973741	0.942755
n=15	1.000000	0.999992	0.999578	0.996018	0.980776	0.953288
n=16	1.000000	0.999996	0.999778	0.997286	0.986460	0.962602
n=17	1.000000	0.999998	0.999872	0.998216	0.990793	0.970697
n=18	1.000000	0.999999	0.999930	0.998864	0.993775	0.977572
n=19	1.000000	0.999999	0.999962	0.999290	0.995406	0.983229
n=20	1.000000	1.000000	0.999979	0.999551	0.996680	0.987666

TABLE 2.6. Reliabilities of linear consecutive-k-out-of-n:G systems (r=0.99)

COMPONENT RELIABILITY = 0.990

	k=2	k=3	k=4	k=5	k=6	k=7
n= 2	0.980100					
n= 3	0.989901	0.970299				
n= 4	0.999702	0.980002	0.960596			
n= 5	0.999897	0.989705	0.970202	0.950990		
n= 6	0.999996	0.999408	0.979808	0.960500	0.941480	
n= 7	0.999999	0.999696	0.989414	0.970010	0.950895	0.932065
n= 8	1.000000	0.999890	0.999020	0.979519	0.960309	0.941386
n= 9	1.000000	0.999990	0.999398	0.989029	0.969724	0.950706
n=10	1.000000	0.999995	0.999684	0.998539	0.979139	0.960027
n=11	1.000000	0.999998	0.999878	0.999005	0.988554	0.969348
n=12	1.000000	0.999999	0.999980	0.999381	0.997968	0.978668
n=13	1.000000	0.999999	0.999989	0.999666	0.998519	0.987989
n=14	1.000000	0.999999	0.999995	0.999861	0.998982	0.997310
n=15	1.000000	0.999999	0.999998	0.999965	0.999355	0.997943
n=16	1.000000	0.999999	0.999999	0.999979	0.999640	0.998489
n=17	1.000000	0.999999	0.999999	0.999988	0.999837	0.998948
n=18	1.000000	0.999999	0.999999	0.999994	0.999944	0.999321
n=19	1.000000	0.999999	0.999999	0.999997	0.999963	0.999607
n=20	1.000000	0.999999	0.999999	0.999999	0.999977	0.999805

III. EVALUATION OF RELIABILITY OF THE CIRCULAR CONSECUTIVE-k-OUT-OF-n:G SYSTEM

A. Introduction

A system with n components, which works whenever at least k consecutive components function, is defined as the consecutive-k-out-of-n:G system. If all components are arranged in a circle, the system is said to be a circular consecutive-k-out-of-n:G system. An example of such circular systems is illustrated in Figure 3.1. Also, the system can be represented by the state diagram of the system as shown in Figure 3.2

It is obvious that for the same k and n, the reliability of the circular system is greater than the reliability of the linear system. For instance, in case k=2, only two end components good cause the linear system fail. However, the circular system can be thought of as the linear system with the two end components closed to form a circle, and in this situation, the system will function if the two end components are good.

B. Notation and Assumptions

n, k, r_i, u_i (i=1,2,...,n) are defined as in Chapter 2.

$R_c(j;k)$ reliability of a circular consecutive-k-out-of-j:G system, j=1,2,...n.

$Q_c(j;k)$ unreliability of the circular system; $Q_c(j;k)=1-R_c(j;k)$.

$R((i,j);k)$ reliability of a linear consecutive-k-out-of-(j-i+1):G

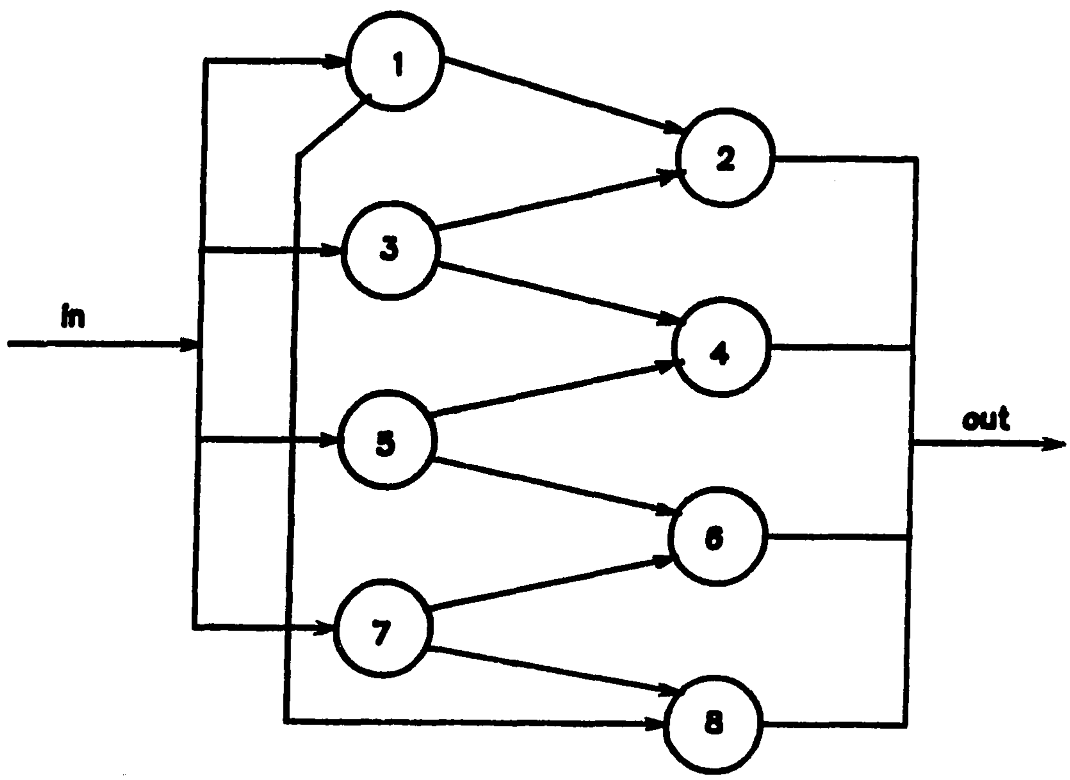

FIGURE 3.1. A circular consecutive-2-out-of-8:G system

subsystem consisting of components i, i+1, ..., j.

Q((i,j);k) unreliability of the linear subsystem;

Q((i,j);k)=1-R((i,j);k).

X_i state of component i.

$$= \begin{cases} 0, & \text{if component } i \text{ fails,} \\ 1, & \text{otherwise.} \end{cases}$$

M random variable indicating the position of the first

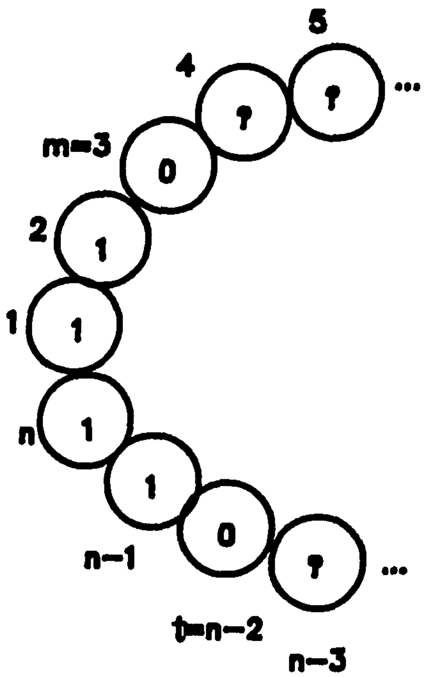

FIGURE 3.2. Scheme of circular system state

failed component clockwise in the circular case;

$M=m$, $m=0,1,2,\ldots,n$.

T random variable indicating the position of the last failed component clockwise; $T=t$, $t=m,m+1,\ldots,n$.

$\{a\}_n$ modulo n

It is assumed that:

- In a system of n components, all components are numbered clockwise in an increasing order.
- Each component is either good or failed.
- $X_1, X_2, \ldots, X_n$ are mutually independent, but not identically distributed unless stated otherwise.

C. System Reliability

To facilitate the derivation process, let us denote $u_{i,j}$ as a vector $(u_i, u_{i+1}, \ldots, u_{j-1}, u_j)$ where $i<j$, and subscripts are reduced by modulo n. From our definitions of M and T, we know that when $m-1+n-t\geq k$, $Q_c(n;k)=0$. Therefore, the system reliability can be obtained by summation of all conditional system unreliabilities over all possible values of m and t subject to $m-1+n-t\geq k$. Thus,

$$Q_c(n;k) = Pr(\text{system fails})$$

$$= \prod_t \prod_m Pr(\text{system fails}/T=t,M=m)\cdot Pr(T=t,M=m)$$

Given m and t that satisfy $m-1+n-t<k$, the conditional system unreliability $Pr(\text{system fails}/T=t,M=m)$ is equivalent to the unreliability of a sub-linear-system with $(t-m-1)$ components. In this subsystem, the component labeled $m+1$ is the first component and the component labeled $t-1$ is the last component of the linear system, i.e.,

$$Pr(\text{system fails}/T=t,M=m) = Q((m+1,t-1);k)$$

Hence, the expression of unreliability for the whole circular system should be as follows:

$$Q_c(n;k) = \sum_{t=n-k+1}^{n} \sum_{m=1}^{\{t+k\}_n} u_m u_t \left(\prod_{i=t+1}^{m-1} r_i \right) Q((m+1,t-1);k)$$

Let $m+1=\ell'$ and $t-1=n'$. Then we can use the formula derived in Chapter 2 to obtain the value of $Q((\ell',n');k)$. The reliability of the circular system follows:

$$R_c(n;k) = 1 - \sum_{t=n-k+1}^{n} \sum_{m=1}^{\{t+k\}_n} u_m u_t \left(\prod_{i=t+1}^{m-1} r_i \right) Q((m+1,t-1);k) \tag{3.1}$$

A computation procedure for the circular consecutive-k-out-of-n:G system, based on the above results, can now be designed. A subroutine to calculate the reliability of the linear consecutive-k-out-of-n:G system is given as follows:

Step 1: Input $R((a,b);k)$ parameter values.

Step 2: Do for $i=a,a+1,\ldots,a+k-2$,

set $R((a,i);k)=0$.

If $b-a+1<k$, set $R((a,b);k)=0$,

go to Step 5.

Step 3: Set $R= \prod_{i=a}^{a+k-1} r_i$,

set $R((a,a+k-1);k)=R$,

If $(b-a+1)=k$, go to Step 5.

Step 4: Do for $i=a+k$ to b,

set $R=R\cdot r_i/r_{i-k}$,

set $R((a,i);k)=R((a,i-1);k)+[1-R((a,i-k-1);k)]u_{i-k}\cdot R$.

Step 5: Output $R((a,b);k)$.

By using the above subroutine procedure to compute the linear system reliability, the procedure to obtain the circular system reliability can be described by the following:

Step 1: Set $Q_c = 0$.

Step 2: Do for $t=n-k+1$ to n,

$$\text{set } R = \prod_{i=t+1}^{n} r_i,$$

do for $m=1$ to $\{t+k\}_n$,

set $R=R\cdot r_{m-1}$,

do $R((m+1,t-1);k)$ subroutine,

set $Q=1-R((m+1,t-1);k)$

set $Q_c=Q_c+R\cdot Q\cdot u_m\cdot u_t$.

Step 3: Set $R_c(n;k) = 1-Q_c$.

Now, let us consider a special case of the circular consecutive-k-out-of-n:G system. In this case, all components have i.i.d. life distributions, i.e., $r_i \equiv r$ for all i's. Define $R((a,b);k)$ with i.i.d. components by $R(b-a+1;k)$. Then, based on formula (3.1) we have:

$$Q_c(n;k)= \sum_{t=n-k+1}^{n} \sum_{m=1}^{\{t+k\}_n} u^2 r^{n-t+m-1} Q(t-m-1;k) \tag{3.2}$$

D. Example

Suppose that n=7, k=3, and the component failure probabilities are the same.

$$R(n;k)=R(n-1;k)+Q(n-k-1;k)ur^k$$

$$=R(n-1;k)+[1-R(n-k-1;k)](1-r)r^k$$

$$R(0;3)=R(1;3)=R(2;3)=0$$

$$R(3;3)=r^3$$

$$R(4;3)=R(3;3)+[1-R(4-3-1;3)](1-r)r^3$$

$$=r^3+[1-R(0;3)](1-r)r^3$$

$$=r^3+(1-r)r^3$$

$$=2r^3-r^4$$

$$R(5;3)=R(4;3)+[1-R(5-3-1;3)](1-r)r^3$$

$$=2r^3-r^4+(1-r)r^3$$

$$=3r^2-2r^4$$

$$R(6;3)=R(5;3)+[1-R(6-3-1;3)](1-r)r^3$$

$$=3r^2-2r^4+(1-r)r^3$$

$$=4r^3-3r^4$$

$$R(7;3)=R(6;3)+[1-R(7-3-1;3)](1-r)r^3$$

$$=4r^3-3r^4+(1-r^3)(1-r)r^3$$

$$=4r^3-3r^4+r^3-r^4-r^6+r^7$$

$$=5r^3-4r^4-r^6+r^7$$

Using formula (3.2), we have

$$Q_C(7;3)=\sum_{t=5}^{7}\sum_{m=1}^{\{t+3\}_n}u^2r^{6-t+m}Q(t-m-1;3)$$

$$=\sum_{m=1}^{1}u^2r^{1+m}Q(4-m;3)$$

$$+\sum_{m=1}^{2}u^2r^{m}Q(5-m;3)$$

$$+\sum_{m=1}^{3}u^2r^{m-1}Q(6-m;3)$$

$$
\begin{aligned}
&=u^2[r^2Q(3;3)+rQ(4;3)+r^2Q(3;3)\\
&\quad +Q(5;3)+rQ(4;3)+r^2Q(3;3)]\\
&=u^2[3r^2Q(3;3)+2rQ(4;3)+Q(5;3)]\\
&=u^2[3r^2(1-r^3)+2r(1-2r^3+r^4)+(1-3r^3+2r^4)]\\
&=(1-r)^2[1+2r+3r^2-3r^3-2r^4-r^5]\\
&=1-7r^3+7r^4-r^7
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
R_C(7;3)&=1-Q_C(7;3)\\
&=7r^3-7r^4+r^7
\end{aligned}
$$

If we enumerate all 2^7 possible system states, 57 out of 128 states guarantee that the circular consecutive-3-out-of-7:G system will function. Therefore,

$$
\begin{aligned}
R_C(7;3)=&7r^3(1-r)^4\\
&+21r^4(1-r)^3\\
&+21r^5(1-r)^2\\
&+7r^6(1-r)\\
&+r^7\\
=&7r^3-28r^4+42r^5-28r^6+7r^7\\
&\quad +21r^4-63r^5+63r^6+21r^7\\
&\quad\quad +21r^5-42r^6+21r^7\\
&\quad\quad\quad +7r^6+7r^7\\
&\quad\quad\quad\quad +7r^7\\
=&7r^3-7r^4+r^7
\end{aligned}
$$

The result obtained by using formula (3.2) matches that by using the above enumeration method.

E. System Reliability Approach 2

A theorem is derived in this section to provide an even better approach to obtaining the reliability of a circular consecutive-k-out-of-n:G system.

Additional Notations:

$R(j;k)$ reliability of a linear consecutive-k-out-of-j:G system, $j=1, \ldots, n$.

$R(r_1,\ldots,r_j;k)$ same as $R(j;k)$, with component reliabilities explicitly expressed by $r_1,\ldots,r_j$.

$R_c(j;k)$ reliability of a circular consecutive-k-out-of-j:G system, $j=1, \ldots, n$.

$R_c(r_1,\ldots,r_j;k)$ same as $R_c(j;k)$, with component reliabilities explicitly expressed by $r_1,\ldots,r_j$.

$Q(r_1,\ldots,r_j;k)$ $1-R(r_1,\ldots,r_j;k)$

Theorem 3.1:

For a circular consecutive-k-out-of-n:G system where all components are not necessarily identical, the system reliability can be obtained from the following equations.

$$R_c(n;k)=0, \quad \text{if } n<k \tag{3.3a}$$

$$R_c(k;k)=\prod_{i=1}^{k} r_i \tag{3.3b}$$

$$R_c(k+1;k)=\prod_{i=1}^{k} r_i+\sum_{i=1}^{k} u_i r_{i+1}\ldots r_n r_1\ldots r_{i-1}$$

$$=\sum_{i=1}^{k+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{k+1} r_j\right)-k\left(\prod_{i=1}^{k+1} r_i\right) \tag{3.3c}$$

$$R_c(n;k)=u_n R(r_1,\ldots r_{n-1};k)+r_n R_c(r_1,\ldots,r_{n-1};k)$$

$$+\sum_{i=1}^{k}(u_{n-k+i-1} r_{n-k+i}\ldots r_n r_1\ldots r_{i-1} u_i)$$

$$\cdot Q(r_{i+1},\ldots,r_{n-k+i-2};k), \quad \text{if } n\geq k+2 \tag{3.3d}$$

Corollary 1:

If all components in the circular consecutive-k-out-of-n:G system are equally reliable, then

$$R_c(n;k)=0, \quad \text{if } n<k \tag{3.4a}$$

$$R_c(k;k)=r^k \tag{3.4b}$$

$$R_c(k+1;k)=r^k+kur^k$$

$$=r^{k+1}+(k+1)ur^k \tag{3.4c}$$

$$R_c(n;k)=uR(n-1;k)+rR_c(n-1;k)$$
$$+ku^2r^kQ(n-k-2;k), \quad \text{if } n\geq k+2 \tag{3.4d}$$

Proof of Theorem 3.1:

It is obvious that equations (3.3a) and (3.3b) hold. We only consider the situations where n>k.

By the pivotal decomposition method,

$$R_c(r_1,\ldots,r_n;k)=u_nR_c(r_1,\ldots,r_{n-1},0;k)+r_nR_c(r_1,\ldots,r_{n-1},1;k) \tag{3.5}$$

and by definition of the circular consecutive-k-out-of-n:G system,

$$R_c(r_1,\ldots,r_{n-1},0;k)=R(r_1,\ldots,r_{n-1};k) \tag{3.6}$$

However, $R_c(r_1,\ldots,r_{n-1},1;k)$ differs for the n=k+1 and n≥k+2 situations.

If n=k+1,

$$R_c(r_1,\ldots,r_k,1;k)=R_c(r_1,\ldots,r_k;k)$$
$$+\sum_{i=1}^{k}u_ir_{i+1}\ldots r_kr_1\ldots r_{i-1} \tag{3.7}$$

Substituting equations (3.6) and (3.7) into equation (3.5), we have

$$R_c(r_1,\ldots,r_{k+1};k)=u_{k+1}R(r_1,\ldots,r_k;k)$$

$$+r_{k+1}[R_c(r_1,\ldots,r_k;k)+\sum_{i=1}^{k} u_i r_{i+1}\ldots r_k r_1\ldots,r_{i-1}]$$

$$=R(k;k)+\sum_{i=1}^{k} u_i r_{i+1}\ldots r_{k+1} r_1\ldots r_{i-1}$$

(3.8)

since $R_c(k;k)=R(k;k)$.

If $n\geq k+2$,

$$R_c(r_1,\ldots,r_{n-1},1;k)=R_c(r_1,\ldots,r_{n-1};k)$$
$$+Pr\{E_1 \text{ and } E_2 \text{ and } E_3\}$$

where E_1 is the event that exactly k-1 components around component n are good, E_2 the event that two components surrounding k consecutive good components (including component n) are failed, and E_3 the event that the remaining n-k-2 components do not comprise any sequence of at least k consecutive good components. Therefore,

$$R_c(r_1,\ldots,r_{n-1},1;k)=R_c(r_1,\ldots,r_{n-1};k)$$

$$+\sum_{i=1}^{k}(u_{n-k+i-1}r_{n-k+i}\ldots r_{n-1}r_1\ldots r_{i-1}u_i)Q(r_{i+1},\ldots,r_{n-k+i-2};k)$$

(3.9)

Substituting equations (3.6) and (3.9) into equation (3.5), we have

$$R_c(r_1,\ldots,r_n;k)$$

$$= u_n R(r_1,\ldots,r_{n-1};k)+r_n[R_c(r_1,\ldots,r_{n-1};k)$$

$$+\sum_{i=1}^{k}(u_{n-k+i-1}r_{n-k+i}\ldots r_{n-1}r_1\ldots r_{i-1})\cdot Q(r_{i+1},\ldots,r_{n-k+i-2};k)]$$

$$= u_n R(r_1,\ldots,r_{n-1};k) + r_n R_c(r_1,\ldots,r_{n-1};k)$$

$$+ \sum_{i=1}^{k} (u_{n-k+i-1} r_{n-k+i} \cdots r_n r_1 \cdots r_{i-1} u_i) \cdot Q(r_{i+1},\ldots,r_{n-k+i-2};k)$$

Q.E.D.

The proof of Corollary 1 automatically follows when all components in the system are equally reliable.

Reliabilities of circular consecutive-k-out-of-n:G systems with equal component reliability r are tabulated in Table 3.1 (r=0.65), Table 3.2 (r=0.8), Table 3.3 (r=0.95) and Table 3.4 (r=0.99).

F. Computation Efficiency

First, let us consider the method of reliability evaluation for a linear consecutive-k-out-of-n:G system. If only multiplications, divisions, additions and subtractions are considered as dominant operations in computation, the procedure described in Section C, to compute reliability of a linear consecutive-k-out-of-n:G system, requires 4n-3k-1 multiplications/divisions, and 2n-2k additions/subtractions. Therefore, this procedure requires o(n) computation time.

Now consider the procedure to evaluate reliability of a circular consecutive-k-out-of-n:G system, described in Section C. The procedure is the implementation of equation (3.1). In this situation, it requires at most k(k+1)(n+2)/2 multiplications and at most k(k+1)+1

TABLE 3.1. Reliabilities of circular consecutive-k-out-of-n:G systems (r=0.65)

COMPONENT RELIABILITY = 0.650

	k=2	k=3	k=4	k=5	k=6	k=7
n= 2	0.422500					
n= 3	0.718250	0.274625				
n= 4	0.770006	0.562981	0.178506			
n= 5	0.855404	0.596623	0.428415	0.116029		
n= 6	0.897067	0.652131	0.450282	0.319080	0.075419	
n= 7	0.931078	0.721853	0.486362	0.333293	0.233798	0.049022
n= 8	0.952460	0.763859	0.531682	0.356745	0.243037	0.169127
n= 9	0.967681	0.802631	0.583006	0.386203	0.258281	0.175132
n=10	0.977873	0.836068	0.618717	0.419564	0.277429	0.185041
n=11	0.984903	0.862803	0.653062	0.455462	0.299113	0.197486
n=12	0.989682	0.885500	0.685153	0.483114	0.322447	0.211581
n=13	0.992954	0.904471	0.714412	0.510190	0.346853	0.226748
n=14	0.995186	0.920228	0.740464	0.536312	0.367078	0.242612
n=15	0.996712	0.933415	0.764286	0.561239	0.387059	0.258929
n=16	0.997754	0.944421	0.785961	0.584810	0.406638	0.273185
n=17	0.998466	0.953603	0.805632	0.606924	0.425711	0.287338
n=18	0.998952	0.961270	0.823475	0.627915	0.444212	0.301320
n=19	0.999284	0.967670	0.839690	0.647806	0.462097	0.315090
n=20	0.999511	0.973013	0.854417	0.666636	0.479338	0.328617

TABLE 3.2. Reliabilities of circular consecutive-k-out-of-n:G systems (r=0.80)

COMPONENT RELIABILITY = 0.800	k=2	k=3	k=4	k=5	k=6	k=7
n= 2	0.640000					
n= 3	0.896000	0.512000				
n= 4	0.921600	0.819200	0.409600			
n= 5	0.967680	0.839680	0.737280	0.327680		
n= 6	0.980992	0.876543	0.753663	0.655359	0.262144	
n= 7	0.991027	0.926515	0.783154	0.668467	0.576716	0.209715
n= 8	0.995164	0.945029	0.823132	0.692060	0.587202	0.503316
n= 9	0.997597	0.961445	0.871497	0.724041	0.606076	0.511705
n=10	0.998745	0.974087	0.893019	0.762734	0.631662	0.526804
n=11	0.999364	0.981612	0.913199	0.806795	0.662615	0.547272
n=12	0.999672	0.987241	0.930963	0.829381	0.697865	0.572035
n=13	0.999833	0.991189	0.945452	0.851108	0.736549	0.600235
n=14	0.999914	0.993843	0.955979	0.871289	0.758742	0.631182
n=15	0.999956	0.995725	0.964743	0.889375	0.780384	0.664329
n=16	0.999977	0.997032	0.971854	0.904924	0.801037	0.685162
n=17	0.999988	0.997934	0.977509	0.917586	0.820348	0.705642
n=18	0.999994	0.998565	0.981978	0.928768	0.838037	0.725489
n=19	0.999997	0.999002	0.985584	0.938525	0.853878	0.744478
n=20	0.999998	0.999306	0.988472	0.946961	0.867690	0.762428

TABLE 3.3. Reliabilities of circular consecutive-k-out-of-n:G systems (r=0.95)

COMPONENT RELIABILITY = 0.950	k=2	k=3	k=4	k=5	k=6	k=7
n= 2	0.902500					
n= 3	0.992750	0.857375				
n= 4	0.995006	0.985981	0.814506			
n= 5	0.999406	0.988124	0.977407	0.773781		
n= 6	0.999733	0.992304	0.979444	0.967226	0.735091	
n= 7	0.999958	0.998418	0.983414	0.969160	0.955619	0.698337
n= 8	0.999985	0.999019	0.989223	0.972933	0.957457	0.942755
n= 9	0.999997	0.999528	0.996777	0.978451	0.961040	0.944501
n=10	0.999999	0.999858	0.997697	0.985627	0.966283	0.947905
n=11	1.000000	0.999926	0.998534	0.994379	0.973100	0.952885
n=12	1.000000	0.999968	0.999210	0.995647	0.981415	0.959362
n=13	1.000000	0.999988	0.999649	0.996840	0.991152	0.967261
n=14	1.000000	0.999994	0.999780	0.997888	0.992783	0.976511
n=15	1.000000	0.999997	0.999874	0.998721	0.994346	0.987044
n=16	1.000000	0.999999	0.999933	0.999277	0.995778	0.989043
n=17	1.000000	0.999999	0.999965	0.999495	0.997018	0.990980
n=18	1.000000	0.999999	0.999980	0.999663	0.998006	0.992799
n=19	1.000000	1.000000	0.999989	0.999785	0.998689	0.994444
n=20	1.000000	1.000000	0.999994	0.999867	0.999014	0.995863

TABLE 3.4. Reliabilities of circular consecutive-k-out-of-n:G systems (r=0.99)

COMPONENT RELIABILITY = 0.990	k=2	k=3	k=4	k=5	k=6	k=7
n= 2	0.980100					
n= 3	0.999702	0.970299				
n= 4	0.999800	0.999408	0.960596			
n= 5	0.999995	0.999505	0.999020	0.950990		
n= 6	0.999998	0.999698	0.999116	0.998539	0.941480	
n= 7	1.000000	0.999986	0.999307	0.998634	0.997969	0.932065
n= 8	1.000000	0.999992	0.999592	0.998824	0.998063	0.997310
n= 9	1.000000	0.999996	0.999970	0.999106	0.998250	0.997403
n=10	1.000000	0.999999	0.999980	0.999481	0.998530	0.997588
n=11	1.000000	1.000000	0.999988	0.999947	0.998901	0.997865
n=12	1.000000	1.000000	0.999995	0.999961	0.999362	0.998232
n=13	1.000000	1.000000	0.999999	0.999974	0.999913	0.998689
n=14	1.000000	1.000000	0.999999	0.999985	0.999932	0.999234
n=15	1.000000	1.000000	0.999999	0.999993	0.999950	0.999868
n=16	1.000000	1.000000	0.999999	0.999998	0.999967	0.999893
n=17	1.000000	1.000000	0.999999	0.999999	0.999980	0.999917
n=18	1.000000	1.000000	0.999999	0.999999	0.999991	0.999939
n=19	1.000000	1.000000	0.999999	0.999999	0.999997	0.999959
n=20	1.000000	1.000000	0.999999	0.999999	0.999997	0.999976

additions/subtractions. As a result, the procedure requires $o(k^2n)$ computation time.

Lastly, let us analyze equation (3.3d). The $Q(r_{i+1},\ldots,r_{n-k+i-2};k)$ takes at most $o(n)$ computation time and $u_{n-k+i-1}r_{n-k+i}\ldots r_{i-1}u_i$ takes $o(k)$ computation time. Then,

$$\prod_{i=1}^{k} (u_{n-k+i-1}r_{n-k+i}\ldots r_{i-1}u_i)Q(r_{i+1},\ldots,r_{n-k+i-2};k)$$

takes $k[o(n)+o(k))]=o(kn)$ computation time. Therefore, computation of equation (3.3d) requires $o(kn)$ time plus the time needed to compute $R_c(r_1,\ldots,{}_{n-1};k)$. If we assume inductively that the computation of $R_c(r_1,\ldots,{}_{n-1};k)$ takes $o(k(n-1))$ time, then the computation of $R_c(r_1,\ldots,{}_n;k)$ requires at most $o(kn)$ time.

IV. BOUNDS ON RELIABILITY OF CONSECUTIVE-k-OUT-OF-n SYSTEMS

This chapter is devoted to establishing bounds on the reliability of consecutive-k-out-of-n systems. In the first section, bounds are developed on the reliability of consecutive-k-out-of-n:G systems introduced in this research, including both the linear systems and the circular systems respectively with the i.i.d. case and the non-i.i.d. case. In the second section, only bounds on the reliability of circular consecutive-k-out-of-n:F systems are analyzed, since the bounds on the reliability of linear systems have been considered by many researchers.

Notation

$R(n;k)$ reliability of a linear consecutive-k-out-of-n:G system.

$Q(n;k)$ unreliability of a linear system; $Q(n;k)=1-R(n;k)$.

$\ell(n;k)$ lower bound on reliability of a linear consecutive-k-out-of-n:G system.

$u(n;k)$ upper bound on reliability of a linear system.

$R_c(n;k)$ reliability of a circular consecutive-k-out-of-n system, either G system or F system.

$Q_c(n;k)$ unreliability of a circular system.

$\ell_c(n;k)$ lower bound on reliability of a circular consecutive-k-out-of-n system, either G system or F system.

$u_c(n;k)$ upper bound on reliability of a circular system.

r_i reliability of component i in a system.

r component reliability in the i.i.d. case.

[a] the largest integer value less than or equal to a.

A. Bounds on Reliability of Consecutive-k-out-of-n:G Systems

A good system should have a high reliability, say at least .99 or even .9999. From a practical viewpoint, the lower bound on the reliability of a system is more important than the upper bound on the system reliability.

1. Bounds on reliability of the linear systems

Suppose that the linear system has n equally reliable components with component reliability r. Function of any k consecutive components supports operation of the system, and any sequence of k consecutive components constitutes a path which is either open or closed. There are n-k+1 such distinct paths in a linear system and the probability that the path is open is r^k. If the system is good, there must be at least one path open. Therefore, the system reliability has an upper bound $1-(1-r^k)^{n-k+1}$, i.e.,

$$R(n;k) \le 1-(1-r^k)^{n-k+1} \tag{4.1a}$$

The lower bound can be determined in the following way. Suppose that a system of n components is divided into two subsystems, one subsystem with n_1 components and the other subsystem with n_2 components ($n=n_1+n_2$). Components 1 through n_1 form a linear consecutive-k-out-of-n_1:G system and components n_1+1 through n a linear consecutive-k-out-of-n_2:G system. 'The n system fails' means that both 'the n_1 system fails' and 'the n_2 system fails', because there does not exist any sequence of k consecutive good components to support 'the n system'. Thus, we have

$$Q(n;k) \le Q(n_1;k)\cdot Q(n_2;k)$$

since $Q(n_1;k)\cdot Q(n_2;k)$ could possibly include some probabilities that 'the n system' is good. In fact, the consecutive-k-out-of-n:G system can be partitioned into either [n/k] subsystems, if n is a multiple of k, or [n/k]+1 subsystems, if n is not a multiple of k. Each subsystem has k consecutive components except possibly the last subsystem. If this last subsystem has less than k components, then the reliability of this subsystem is zero and the failure probability is one. Therefore, in general we have

$$Q(n;k) \le \{Q(k;k)\}^{[n/k]}$$
$$= (1-r^k)^{[n/k]}$$

i.e.,

$$R(n;k) \ge 1-(1-r^k)^{[n/k]} \qquad (4.1b)$$

In summary, for a linear consecutive-k-out-of-n:G system with n equally reliable components, the system reliability is bounded by

$$1-(1-r^k)^{[n/k]} \le R(n;k) \le 1-(1-r^k)^{n-k+1} \tag{4.1c}$$

The reliability of the linear consecutive-4-out-of-13:G system is computed and given in Table 4.1. The corresponding lower bound and upper bound are also shown in the table with different values of equal component reliability r. The results for the linear consecutive-4-out-of-17:G system are given in Table 4.2.

From these tables, it can be seen that the higher the system reliability is, the narrower the bound interval achieves.

If all n components in the system are not equally reliable, the above conclusion can be extended to the following:

$$1-\prod_{j=0}^{[n/k]-1}\left(1-\prod_{i=jk+1}^{jk+k} r_i\right) \le R(n;k) \le 1-\prod_{j=1}^{n-k+1}\left(1-\prod_{i=0}^{k-1} r_{j+i}\right) \tag{4.2}$$

2. Bounds on reliability of the circular systems

In the circular system with n equally reliable components, there are no end components. As a result, there are n distinct paths each of which will support the system if and only if all k consecutive components in the path are good. The circular system functions as long as at least one of n distinct paths is open. Therefore, the upper bound on the reliability of the circular consecutive-k-out-of-n:G

TABLE 4.1. Bounds and reliability of the linear consecutive-4-out-of-13:G system

r	$\ell(13;4)$	R(13;4)	u(13;4)
0.15	0.00151795	0.00437611	0.00505090
0.20	0.00479233	0.01309334	0.01588541
0.25	0.01167297	0.03013031	0.03838283
0.30	0.02410370	0.05857868	0.07811052
0.35	0.04434645	0.10110915	0.14032388
0.40	0.07485044	0.15951401	0.22843391
0.45	0.11804283	0.23427552	0.34210241
0.50	0.17602497	0.32421780	0.47553879
0.55	0.25016409	0.42630905	0.61698061
0.60	0.34058738	0.53569227	0.75043094
0.65	0.44561207	0.64600235	0.86002654
0.70	0.56119579	0.75001305	0.93579495
0.75	0.68055499	0.84060723	0.97771633
0.80	0.79420155	0.91199785	0.99485397
0.85	0.89078772	0.96103561	0.99937737
0.90	0.95932716	0.98830914	0.99997687
0.95	0.99361759	0.99859321	1.00000000

TABLE 4.2. Bounds and reliability of the linear consecutive-4-out-of-17:G system

r	$\ell(17;4)$	R(17;4)	u(17;4)
0.15	0.00202340	0.00609167	0.00706410
0.20	0.00638467	0.01816258	0.02216870
0.25	0.01553363	0.04158068	0.05332023
0.30	0.03200847	0.08024329	0.10761881
0.35	0.05868721	0.13708961	0.19077593
0.40	0.09853417	0.21334338	0.30445951
0.45	0.15420848	0.30788791	0.44355583
0.50	0.22752333	0.41692996	0.59486598
0.55	0.31877851	0.53411186	0.73907900
0.60	0.42604703	0.65118229	0.85675913
0.65	0.54457343	0.75921905	0.93625236
0.70	0.66655231	0.85025352	0.97859102
0.75	0.78162909	0.91896451	0.99513394
0.80	0.87849635	0.96396273	0.99937475
0.85	0.94779706	0.98814344	0.99996752
0.90	0.98601252	0.99770820	0.99999970
0.95	0.99881613	0.99987239	1.00000000

system is $1-(1-r^k)^n$. To obtain the lower bound on reliability, the whole system of n components is partitioned into either [n/k] or [n/k]+1 subsystems where each subsystem has k consecutive components except possibly the last one which has n-k[n/k] consecutive components. In the circular system, although this last subsystem has less than k consecutive components, it can still combine with some of the front consecutive components in the first subsystem of k components to support the whole system. A subsystem (i.e., a consecutive-k-out-of-k:G system) functions if and only if all k components are good. Thus, we have

$$Q_C(n;k) \le \{Pr(\text{k component system fails})\}^{[(n+k-1)/k]}$$
$$= (1-r^k)^{[(n+k-1)/k]}$$

or

$$R_C(n;k) \ge 1-(1-r^k)^{[(n+k-1)/k]} \tag{4.3a}$$

In summary, the bounds on reliability of a circular consecutive-k-out-of-n:G system with equally reliable components are

$$1-(1-r^k)^{[(n+k-1)/k]} \le R_C(n;k) \le 1-(1-r^k)^n \tag{4.3b}$$

Table 4.3 shows the reliability and the bounds on reliability of the circular consecutive-4-out-of-13:G system and Table 4.4 is for the circular consecutive-4-out-of-17:G system.

In the case that all components in the circular system are not equally reliable, the bounds on reliability of the circular consecutive-k-out-of-n:G system are

TABLE 4.3. Bounds and reliability of the circular consecutive-4-out-of-13:G system

r	$\ell_C(13;4)$	$R_C(13;4)$	$u_C(13;4)$
0.15	0.00202340	0.00558924	0.00656116
0.20	0.00638467	0.01659737	0.02060163
0.25	0.01553363	0.03786271	0.04960781
0.30	0.03200847	0.07287401	0.10033149
0.35	0.05868721	0.12432975	0.17844748
0.40	0.09853417	0.19355190	0.28618598
0.45	0.15420848	0.27999973	0.41976255
0.50	0.22752333	0.38098049	0.56785709
0.55	0.31877851	0.49164575	0.71279830
0.60	0.42604703	0.60535258	0.83543104
0.65	0.54457343	0.71441066	0.92240041
0.70	0.66655231	0.81118107	0.97182661
0.75	0.78162909	0.88939309	0.99288160
0.80	0.87849635	0.94545162	0.99894100
0.85	0.94779706	0.97940999	0.99993205
0.90	0.98601252	0.99519461	0.99999911
0.95	0.99881613	0.99964857	1.00000000

TABLE 4.4. Bounds and reliability of the circular consecutive-4-out-of-17:G system

r	$\ell_c(17;4)$	$R_c(17;4)$	$u_c(17;4)$
0.15	0.00252861	0.00730270	0.00857133
0.20	0.00797445	0.02164861	0.02685475
0.25	0.01937920	0.04922178	0.06437081
0.30	0.03984922	0.09420961	0.12912858
0.35	0.07281274	0.15938020	0.22666216
0.40	0.12161165	0.24519926	0.35652107
0.45	0.18889111	0.34920996	0.50924009
0.50	0.27580303	0.46588778	0.66617972
0.55	0.38111436	0.58713222	0.80435205
0.60	0.50043112	0.70344222	0.90554518
0.65	0.62586963	0.80563122	0.96465909
0.70	0.74661285	0.88673544	0.99060565
0.75	0.85072279	0.94359094	0.99844557
0.80	0.92826408	0.97750878	0.99987137
0.85	0.97504723	0.99364060	0.99999648
0.90	0.99518973	0.99902624	1.00000000
0.95	0.99978042	0.99996537	1.00000000

$$1-\prod_{i=0}^{[(n+k-1)/k]-1}(1-\prod_{j=ik+1}^{ik+k} r_i) \le R_c(n;k) \le 1-\prod_{i=1}^{n}(1-\prod_{j=0}^{k-1} r_{i+j})$$

(4.4)

B. Bounds on Reliability of a Circular Consecutive-k-out-of-n:F System

A circular consecutive-k-out-of-n:F system contains an ordered sequence of n components on a circle such that the system fails if and only if at least k consecutive components fail. In the system of n equally reliable components, any failed k consecutive components constitute a minimal cut set and would cause the failure of the system. There are n such minimal cut sets in a circular consecutive-k-out-of-n:F system. If the system is good, there must be at least one good component in every cut set. This leads to the lower bound:

$$\{1-(1-r)^k\}^n \le R_c(n;k)$$

(4.5a)

A system of n components can be partitioned into either [n/k] subsystems or [n/k]+1 subsystems, depending on whether n is a multiple of k or not. In the case that n is not a multiple of k, there are [n/k]+1 subsystems with the last one having less than k consecutive components. In either case, each subsystem except possibly the last one has exactly k consecutive components and fails if and only if all k components fail. This leads to a upper bound:

$$R_c(n;k) \le \{1-(1-r)^k\}^{[(n+k-1)/k]}$$

(4.5b)

In summary, the bounds on the reliability of a circular consecutive-k-out-of-n:F system with equal component reliabilities are:

$$\{1-(1-r)^k\}^n \le R_c(n;k) \le \{1-(1-r)^k\}^{[(n+k-1)/k]} \tag{4.5c}$$

Table 4.5 gives the reliability and the bounds on reliability of the circular consecutive-4-out-of-13:G system and Table 4.6 gives the results for the circular consecutive-4-out-of-17:G system.

If all components in the system have the same component reliability, then the bounds on the system reliability are

$$\prod_{i=1}^{n}\left(1-\prod_{j=0}^{k-1}(1-r_{i+j})\right) \le R_c(n;k) \le \prod_{i=0}^{[(n-1)/k]}\left(1-\prod_{j=ik+1}^{j=ik+1}(1-r_{i+j})\right) \tag{4.6}$$

TABLE 4.5. Bounds and reliability of the circular consecutive-4-out-of-13:F system

r	$l_c(13;4)$	$R_c(13;4)$	$u_c(13;4)$
0.15	0.00006800	0.02058937	0.05220217
0.20	0.00105901	0.05454753	0.12150240
0.25	0.00711824	0.11060584	0.21836936
0.30	0.02817295	0.18881720	0.33344591
0.35	0.07759857	0.28558731	0.45542473
0.40	0.16456753	0.39464551	0.57395142
0.45	0.28719968	0.50835234	0.68122000
0.50	0.43214089	0.61901772	0.77247554
0.55	0.58023530	0.71999854	0.84579057
0.60	0.71381235	0.80644679	0.90146518
0.65	0.82155150	0.87566918	0.94131243
0.70	0.89966780	0.92712522	0.96799129
0.75	0.95039201	0.96213675	0.98446631
0.80	0.97939837	0.98340231	0.99361533
0.85	0.99343884	0.99441057	0.99797660
0.90	0.99870104	0.99883008	0.99960017
0.95	0.99991941	0.99992275	0.99997520

TABLE 4.6. Bounds and reliability of the circular consecutive-4-out-of-17:F system

r	$\ell_C(17;4)$	$R_C(17;4)$	$u_C(17;4)$
0.15	0.00000355	0.00635896	0.02495231
0.20	0.00012867	0.02249062	0.07173496
0.25	0.00155441	0.05640823	0.14927584
0.30	0.00939415	0.11326295	0.25338548
0.35	0.03534031	0.19436681	0.37412846
0.40	0.09445369	0.29655576	0.49956721
0.45	0.19564617	0.41286564	0.61888397
0.50	0.33381820	0.53410995	0.72419566
0.55	0.49075758	0.65078783	0.81110770
0.60	0.64347690	0.75479895	0.87838757
0.65	0.77333659	0.84061831	0.92718679
0.70	0.87087059	0.90578926	0.96015048
0.75	0.93562889	0.95077747	0.98062074
0.80	0.97314525	0.97835088	0.99202555
0.85	0.99142867	0.99269694	0.99747139
0.90	0.99830168	0.99847031	0.99950022
0.95	0.99989462	0.99989891	0.99996901

C. Analysis of Bounds on System Reliability

Bounds on the reliability of a system are a function of system size n, the minimal number of consecutive components required for the system to function or fail, and the component reliability r_i's.

Consider the situation where all components in the system are equally reliable, i.e., $r_i=r$ for all i's. Here we use the interval between the upper and lower bounds on system reliability to measure how good the bounds are.

For a linear consecutive-k-out-of-n:G system with equally reliable components, the interval is

$$(1-r^k)^m - (1-r^k)^{n-k+1}$$

where m is the largest integer less than or equal to n/k. The first derivative of the interval with respect to component reliability r is

$$\frac{d}{dr}((1-r^k)^m - (1-r^k)^{n-k+1}) =$$

$$kr^{k-1}[(n-k+1)(1-r^k)^{n-k} - m(1-r^k)^{m-1}]$$

Several figures have been constructed to analyze changes of the bound interval with respect to k, n and r, respectively. In the figures, L-G and C-G represent the curves of reliability intervals for linear G systems and circular G systems respectively, and L-F and C-F the curves for linear F systems and circular F systems respectively. Figures 1 and 2 show that for given values of k and r, as the system size n increases, the bound interval of reliability of consecutive-k-out-of-n:G systems narrows; but the bound interval for consecutive-k-out-of-n:F systems widens. In addition, the choice of k value is critical. In the figures, different k values lead to quite different intervals.

Figures 3 and 4 illustrate the fact that for given values of n and k, intervals are narrower in one range of component reliability, and wider in the other range of component reliability. From Figures 3 and 4, it appears that when k increases, the wider bound intervals of reliability of consecutive-k-out-of-n:G systems move in the direction as the increases of component reliability. In contrast, the wider intervals for consecutive-k-out-of-n:F systems move in the opposite direction. In general, when k is small, the bound intervals for G systems are narrow corresponding to high component reliability.

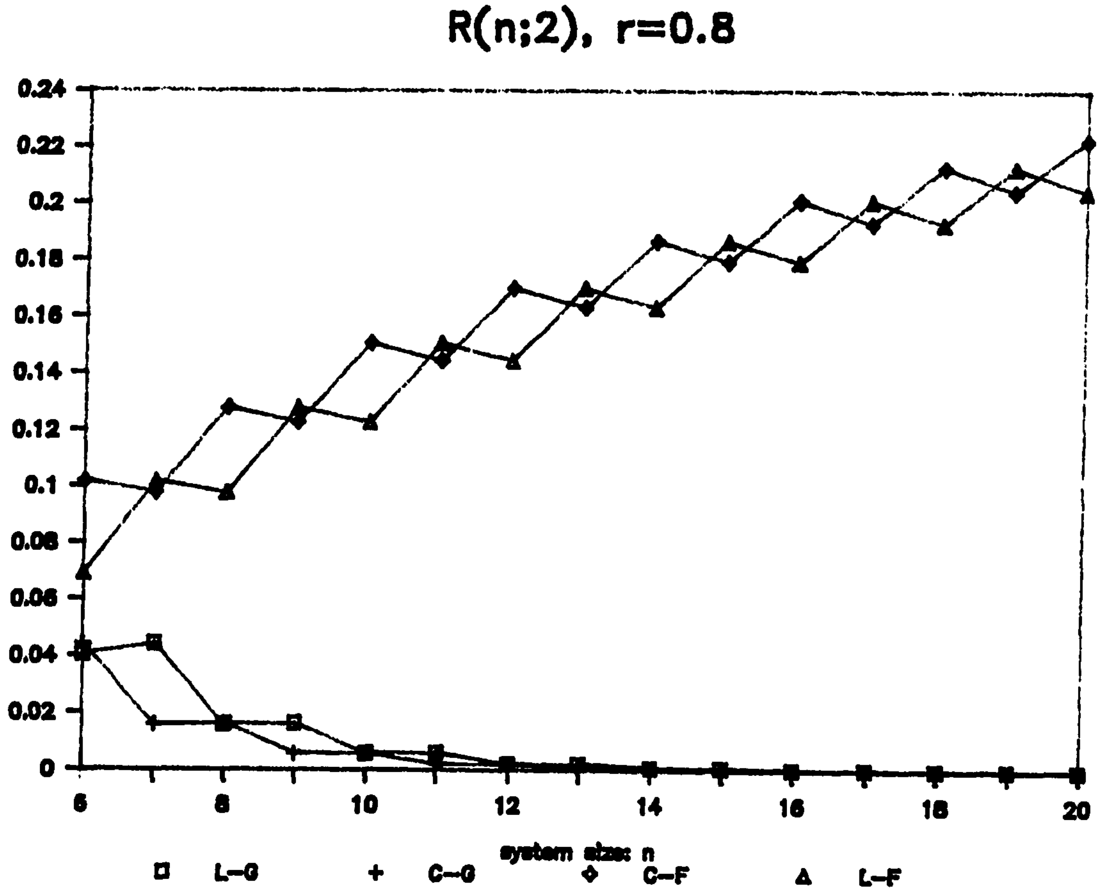

FIGURE 4.1. Bound intervals of system reliabilities as a function of n (k=2, r=0.8)

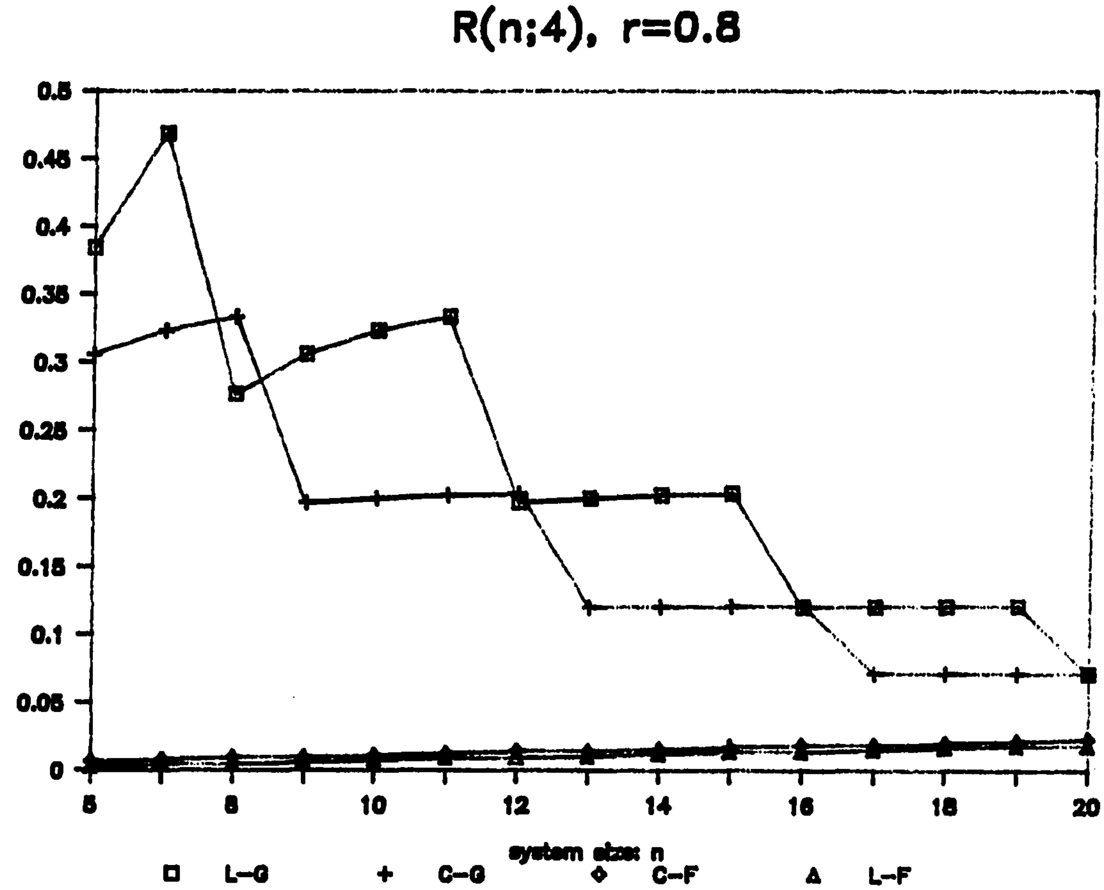

FIGURE 4.2. Bound intervals of system reliabilities as a function of n (k=4, r=0.8)

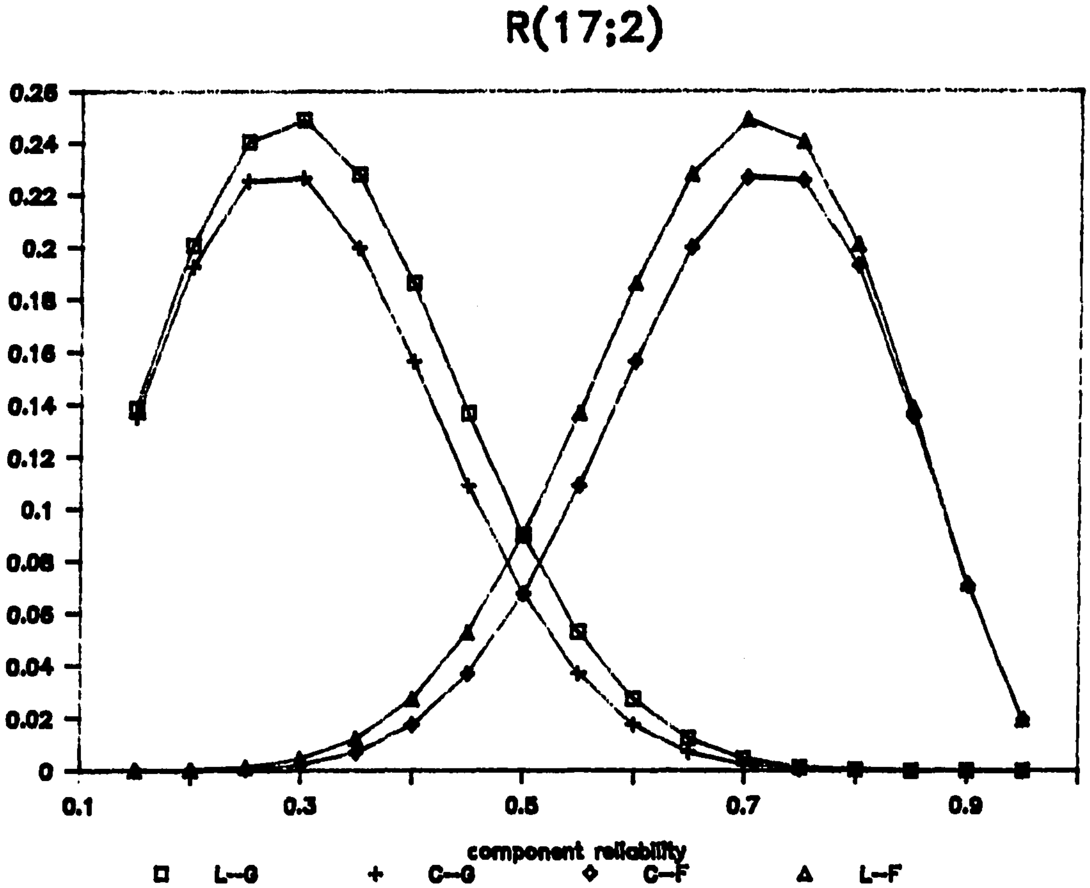

FIGURE 4.3. Bound intervals of system reliabilities as a function of r (n=17, k=2)

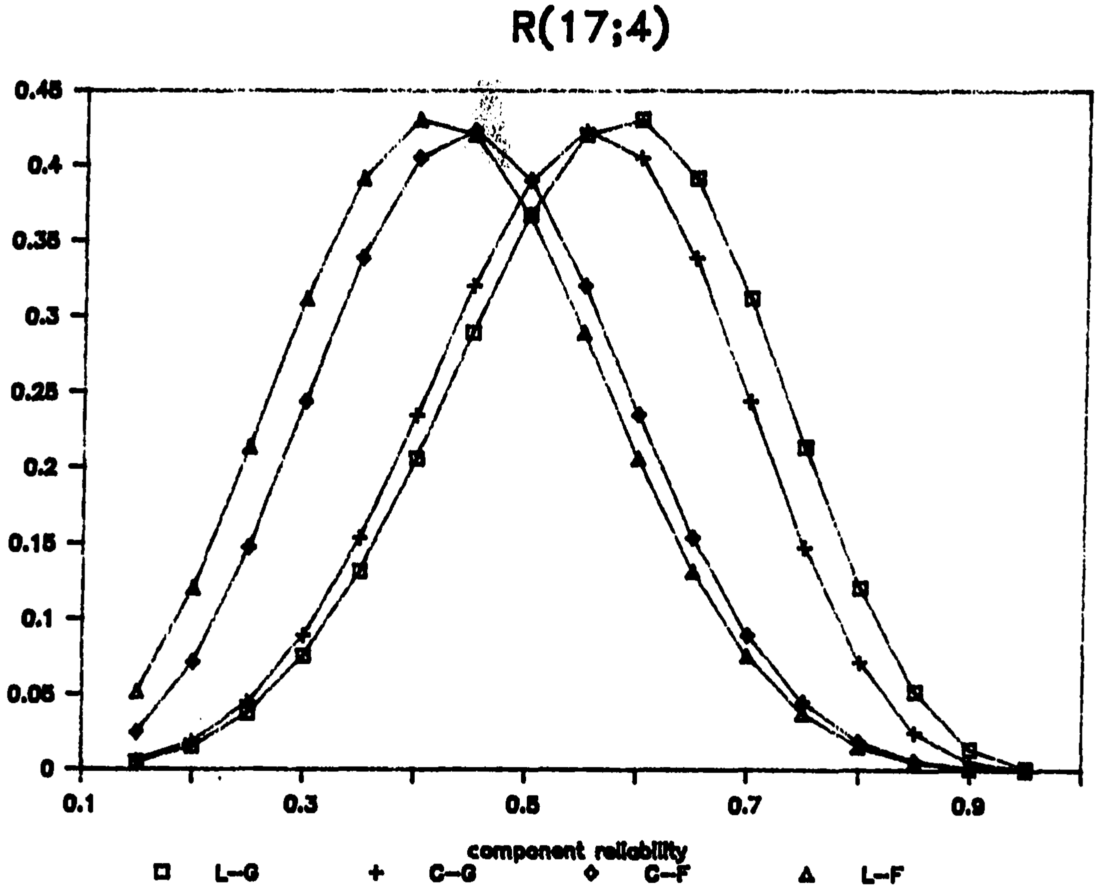

FIGURE 4.4. Bound intervals of system reliabilities as a function of r (n=17, k=4)

V. RELIABILITY IMPORTANCE OF COMPONENTS

In a system whose performance depends on the performance of its components, some components are more important than others. The component importance is fundamental in reliability theory and a number of different measures of component importance have been proposed in the literature [2]. Different perspectives on the same system can lead to different views on the factors which make one component play a more important role than others. Here, we use the measure of the reliability importance of each component which takes into account component reliability as well as system structure. This is very useful in analyzing the effect that an improvement in a particular component will make on system reliability, and permits the analyst to determine those components on which additional research and development effort can be more profitably expended. The concept of "reliability importance" is originally due to Birnbaum. Therefore, reliability importance is an equivalent name for the B-importance (Birnbaum importance).

The following are the notations to be used in this chapter.

n, k, r_i, u_i, r and u are defined as before.

$R(j;k)$ reliability of a linear consecutive-k-out-of-j:G subsystem consisting of components $1,2,\ldots,j$.

$R(r_1,r_2,\ldots,r_j;k)$ same as $R(j;k)$, with component reliabilities explicitly expressed.

$R'(j;k)$ reliability of a linear consecutive-k-out-of-j:G subsystem consisting of components n-j+1, n-j+2, ..., n-1, n.

$R(r_{n-j+1}, r_{n-j+2}, \ldots, r_n; k)$ same as $R'(j;k)$, with explicit component reliabilities.

$R_c(n;k)$ reliability of a circular consecutive-k-out-of-n:G system.

$R_i(n-1;k)$ reliability of a linear consecutive-k-out-of-(n-1):G system consisting of components i+1, i+2, ..., n, 1, 2, ..., i-1, for i=1,2,...,n, i.e., the system contains all components except component i.

$R_i(r_{i+1}, \ldots, r_n, r_1, \ldots, r_{i-1}; k)$ same as $R_i(n-1;k)$.

I_i B-importance of component i.

A. Component Importance in a Linear Consecutive-k-out-of-n:G System

The reliability importance of component i is defined by

$$I_i = \frac{\partial R(n;k)}{\partial r_i}$$

$$= R(r_1, \ldots, r_{i-1}, 1, r_{i+1}, \ldots, r_n; k) - R(r_1, \ldots, r_{i-1}, 0, r_{i+1}, \ldots, r_n; k) \qquad (5.1)$$

I_i measures the changing rate of the system reliability with respect to the reliability of component i. It is also the decrease in system reliability when component i fails.

Theorem 5.1:

For a linear consecutive-k-out-of-n:G system with unequally reliable components, we have

$$I_i = (1/r_i)[R(n;k)-R(i-1;k)-R'(n-i;k)+R(i-1;k)R'(n-i;k)] \tag{5.2}$$

Corollary 1:

In a linear consecutive-k-out-of-n:G system with the i.i.d. components ($r_1=r_2=\ldots=r_n$), if $n\geq 2k+1$, then the reliability importance of components increases from component 1 through component k, and decreases from component n-k+1 through component n. If $n<2k-1$ (i.e., $n-k+1<k$), then the component importance increases from component 1 through component n-k+1, and decreases from component k to component n; all components in between component n-k and component k+1 have the same component importance:

$$I_i = R(n;k)/r, \quad \text{for } i = n-k+1, \ldots, k. \tag{5.3}$$

Corollary 2:

If $n<2k-1$, then the component with the smallest reliability in between components n-k and k+1 is the most important one in this range.

Proof of Theorem 5.1:

From the pivotal decomposition theorem, we have:

$$R(n;k) = r_i R(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_n;k) + u_i R(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k) \tag{5.4}$$

Then,

$$R(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_n;k) = (1/r_i)[R(n;k)-u_i R(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k)] \tag{5.5}$$

By definition of a linear consecutive-k-out-of-n:G system,

$$\begin{aligned} R(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k) &= R(r_1,\ldots,r_{i-1};k)+R(r_{i+1},\ldots,r_n;k) \\ &\quad - R(r_1,\ldots,r_{i-1};k)R(r_{i+1},\ldots,r_n;k) \\ &= R(i-1;k)+R'(n-i;k)-R(i-1;k)R'(n-i;k) \end{aligned} \tag{5.6}$$

Substituting equations (5.5) and (5.6) into equation (5.2), we obtain

$$\begin{aligned} I_i &= (1/r_i)[R(n;k)-u_i R(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k)] \\ &\quad -R(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k) \\ &= (1/r_i)[R(n;k)-R(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k)] \\ &= (1/r_i)[R(n;k)-R(i-1;k)-R'(n-i;k)+R(i-1;k)R'(n-i;k)] \end{aligned}$$

Q.E.D.

Proof of Corollary 1:

Note that in the i.i.d. case, $R'(j;k)=R(j;k)$. If $n=2k-1$, then $k=n-k+1$. From equation (5.3), it is obvious that if $i \le k$, then

$R(i-1;k)=0$ and $R(n-i;k)$ decrease as i increases (up to k). Therefore, I_i increases as i increases. If $i \geq n-k+1=k$, then $R(n-i;k)=0$ and $R(i-1;k)$ increase as i increases. Therefore, I_i decreases as i increases (up to n).

If $n>2k-1$, then $k>n-k+1$. By the same token, I_i increases from component 1 to component k and decreases from component n-k+1 to component n.

If $n<2k-1$, then $n-k+1<k$. From equation (5.2), for $n-k<i<k+1$,

$$R(i-1;k) = R'(n-i;k) = R(n-i;k) = 0$$

Therefore,

$$I_i = (1/r)R(n;k), \text{ for } n-k < i < k+1$$

i.e., all i.i.d. components in this range are equally important.

Q.E.D.

Proof of Corollary 2:

If $n<2k+1$, then $n-k+1<k$. From equation (5.2), for $n-k<i<k+1$, we have

$$R(i-1;k) = R'(n-i;k) = 0$$

Therefore,

$$I_i = (1/r_i)R(n;k), \text{ for } n-k < i < k+1$$

If r_i is the smallest value in this range, then I_i will be the largest in the range.

Q.E.D.

When $n>2k$ in the i.i.d. case, the reliability importance of the components in between component k and component n-k+1 may not be the same. This contradicts the intuitive conjecture that all of these components should have had the same reliability importance in the i.i.d. case because they have the same number of k-tuples. The following process will clarify why this is not true.

Assume that $j=i+1\leq n/2$.

$$
\begin{aligned}
I_j-I_i=&(1/r)[R(n;k)-R(j-1;k)-R(n-j;k)+R(j-1;k)R(n-j;k)]\\
&-(1/r)[R(n;k)-R(i-1;k)-R(n-i;k)+R(i-1;k)R(n-i;k)]\\
=&(1/r)[R(i-1;k)+R(n-i;k)+R(j-1;k)R(n-j;k)\\
&-R(j-1;k)-R(n-j;k)-R(i-1;k)R(n-i;k)]\\
=&(1/r)[R(i-1;k)+R(n-i;k)+R(i;k)R(n-i-1;k)\\
&-R(i;k)-R(n-i-1;k)-R(i-1;k)R(n-i;k)]
\end{aligned}
$$

By using equation (2.3),

$$
\begin{aligned}
R(n;k) &= R(n-1;k)+ur^k[1-R(n-k-1;k)]\\
&= R(n-1;k)+ur^k-ur^kR(n-k-1;k)
\end{aligned}
$$

we have

$$R(i;k) = R(i-1;k)+ur^k-ur^kR(i-k-1;k)$$

$$R(n-i;k) = R(n-i-1;k)+ur^k-ur^kR(n-i-k-1;k)$$

Therefore,

$$
\begin{aligned}
I_j-I_i=(1/r)[&R(i-1;k)+R(n-i-1;k)+ur^k-ur^kR(n-i-k-1;k)\\
&+R(i-1;k)R(n-i-1;k)+ur^kR(n-i-1;k)\\
&-ur^kR(i-k-1;k)R(n-i-1;k)\\
&-R(i-1;k)-ur^k+ur^kR(i-k-1;k)-R(n-i-1;k)\\
&-R(i-1;k)R(n-i-1;k)-ur^kR(i-1;k)
\end{aligned}
$$

$$+ur^kR(i-1;k)R(n-i-k-1;k)]$$

$$=(1/r)[R(n-i-1;k)+R(i-k-1;k)+R(i-1;k)R(n-i-k-1;k)$$

$$-R(n-i-k-1;k)-R(i-1;k)-R(i-k-1;k)R(n-i-1;k)]$$

Although $R(n-i-1;k)>R(n-i-k-1;k)$, but $R(i-k-1;k)<R(i-1;k)$. In addition, usually $R(i-1;k)R(n-i-k-1;k)>R(i-k-1;k)R(n-i-1;k)$, since

$$(i-1)+(n-i-k-1)=(i-k-1)+(n-i-1)=n-k-2$$

and

$$|(n-i-k-1)-(i-1)|=|n-2i-k|$$

$$< |(n-i-1)-(i-k-1)|=|n-2i+k|$$

The critical point is that $R(j;k)$ is not linear and therefore, we do not have a general rule to determine which component is more important than the others among the n-2k components in between component k and component n-k+1.

B. Component Importance in a Circular Consecutive-k-out-of-n:G System

The reliability importance of component i in a circular consecutive-k-out-of-n:G system is defined as

$$I_i = R_c(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_n;k)$$
$$-R_c(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k) \qquad (5.7)$$

Theorem 5.2:

For a circular consecutive-k-out-of-n:G system with unequally reliable components, the reliability importance of component i is

$$I_i = (1/r_i)[R_c(n;k)-R_i(n-1;k)]$$

(5.8)

Corollary 1:

For a circular consecutive-k-out-of-n:G system with the i.i.d. components, all components have the same reliability importance, i.e., $I_1=I_2=\ldots=I_n=I=(1/r)[R_c(n;k)-R(n-1;k)]$

(5.9)

Proof of Theorem 5.2:

By using the pivotal decomposition theorem, we have

$$R_c(n;k) = r_iR_c(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_n;k)$$
$$+ u_iR_c(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k)$$

(5.10)

Then,

$$R_c(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_n;k)$$
$$= (1/r_i)[R_c(n;k)-q_iR_c(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k)]$$

(5.11)

By definition of a circular consecutive-k-out-of-n:G system,

$$R_c(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k)$$
$$= R(r_{i+1},\ldots,r_n,r_1,\ldots,r_{i-1};k)$$
$$= R_i(n-1;k)$$

(5.12)

Substituting equations (5.11) and (5.12) into definition (5.7), we have

$$I_i = (1/r_1)[R_c(n;k)-u_iR_c(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k)]$$
$$-R_c(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k)$$
$$= (1/r_i)[R_c(n;k)-R_c(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_n;k)]$$

$= (1/r_i)[R_c(n;k)-R_i(n-1;k)]$

Q.E.D.

Proof of Corollary 1:

If all components have the i.i.d. failure distributions, i.e., $r_i=r$ and $R_i(n-1;k)=R(n-1;k)$ for $i=1,2,\ldots,n$, then it is obvious that

$$I_1=I_2=\ldots=I_n=I=(1/r)[R_c(n;k)-R(n-1;k)]$$

Q.E.D.

C. Examples

Two examples of component importance in the linear consecutive-k-out-of-n:G systems are provided in this section. One is an example of a system with n > 2k and the other with n < 2k.

First, consider a linear consecutive-3-out-of-13:G system. All components in the system are i.i.d. with r=0.5. By equations (2.1) and (5.2), system reliability R(i;k) and reliability importance of component i for i=1, 2, ..., 13 are given in Table 5.1.

From the results in Table 5.1, we see that components k and n-k+1 are the most important and the two end components are the least important. The components in between component k and component n-k+1 have about the same importance. Therefore, if we want to expend additional effort to improve the overall system reliability, components k and n-k+1 should be considered first.

TABLE 5.1. Reliability importance and system reliability in a linear consecutive-3-out-of-13:G system

i	$R(i;3)$	I_i
1	0.00000000	0.06689453
2	0.00000000	0.13964844
3	0.12500000	0.21875000
4	0.18750000	0.17089844
5	0.25000000	0.18017578
6	0.31250000	0.18359375
7	0.36718750	0.17968750
8	0.41796875	0.18359375
9	0.46484375	0.18017578
10	0.50781250	0.17089844
11	0.54736328	0.21875000
12	0.58374023	0.13964844
13	0.61718750	0.06689453

Now, let us consider a linear consecutive-3-out-of-14:G system with equally reliable components (r=0.5). The results of interest are given in Table 5.2. Components in between positions n-k and k+1 are equally important. In fact, failure of any component in this range will cause the system fail.

Figures 5.1 and 5.2 show the distribution of component importance in a linear consecutive-3-out-of-13:G system and a linear consecutive-9-out-of-13:G system, respectively, with equal component reliability r=0.5.

TABLE 5.2. Reliability importance and system reliability in a linear consecutive-9-out-of-13:G system

i	R(i;9)	I_i
1	0.00000000	0.00195312
2	0.00000000	0.00390625
3	0.00000000	0.00585937
4	0.00000000	0.00781250
5	0.00000000	0.01171875
6	0.00000000	0.01171875
7	0.00000000	0.01171875
8	0.00000000	0.01171875
9	0.00195312	0.01171875
10	0.00292969	0.00781250
11	0.00390625	0.00585937
12	0.00488281	0.00390625
13	0.00585937	0.00195312

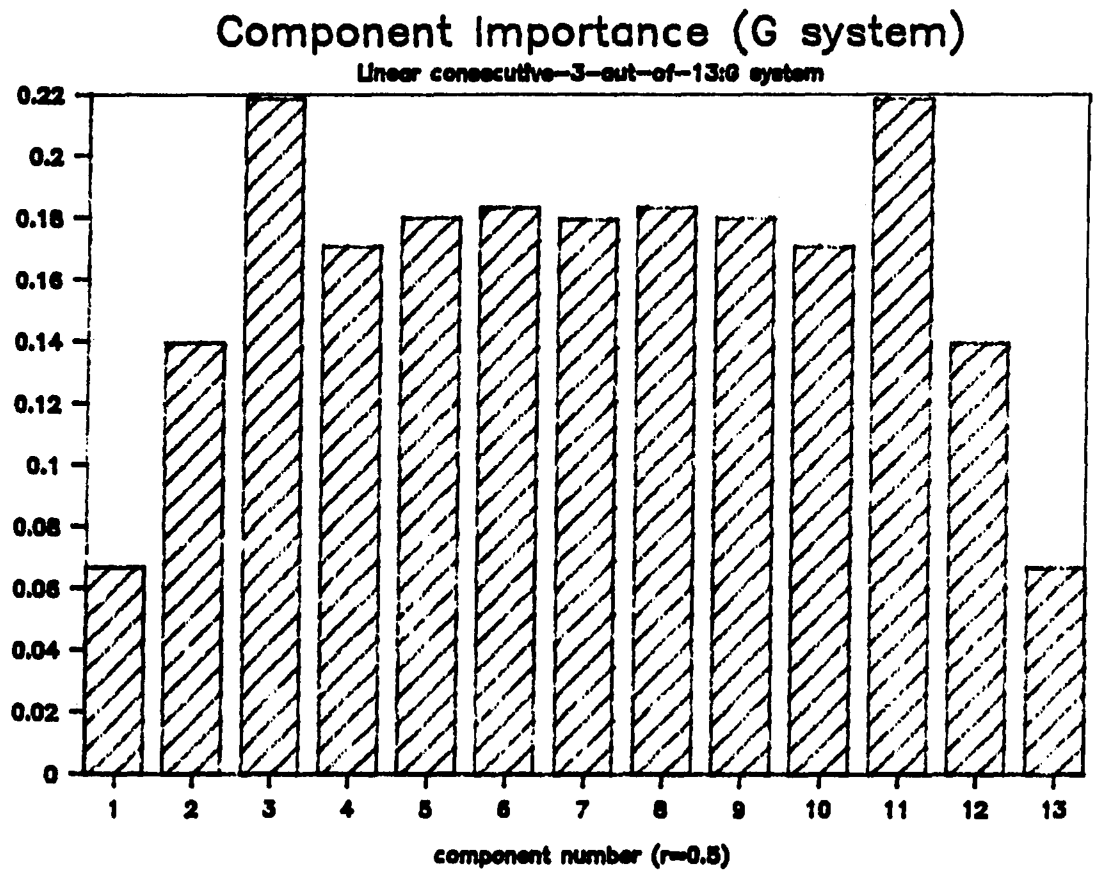

FIGURE 5.1. Component importance of a linear consecutive-3-out-of-13:G system

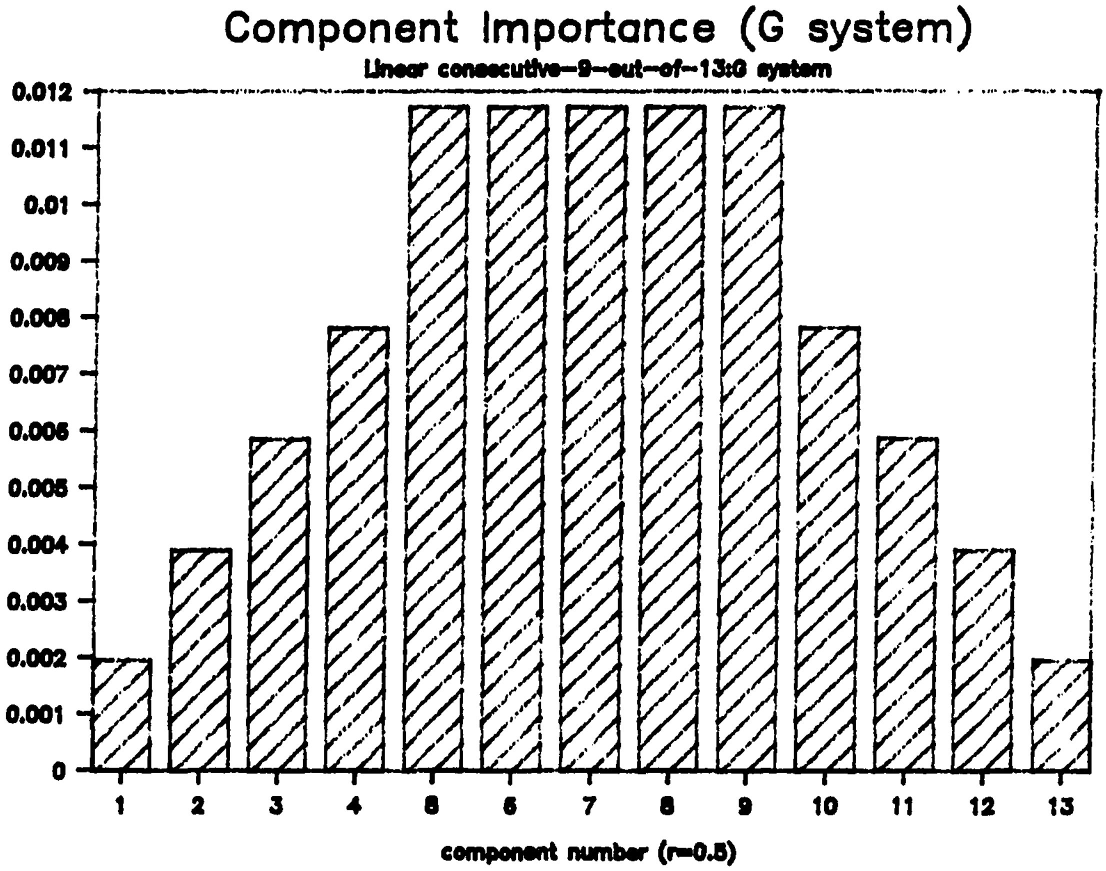

FIGURE 5.2. Component importance of a linear consecutive-9-out-of-13:G system

VI. RELATIONS BETWEEN CONSECUTIVE-k-OUT-OF-n SYSTEMS

The consecutive-k-out-of-n:F systems have been studied for several years. This study introduces the concept of consecutive-k-out-of-n:G systems. An interesting question is whether or not there exist any relations between these two types of the systems. This chapter is to investigate if some connections exist between the two systems.

A. Relation Between Consecutive-k-out-of-n:F Systems and Consecutive-k-out-of-n:G Systems

There are two cases for the consecutive-k-out-of-n:F systems. One is the linear case and the other is the circular case. Similarly, this applies to the consecutive-k-out-of-n:G systems. The following theorem will provide a way of using the existing methods for one type of consecutive-k-out-of-n system to search for a solution to the other type of consecutive-k-out-of-n system.

Theorem 6.1:

If the reliability of component i, r_i, in one type of consecutive-k-out-of-n system (say F system) is equal to the unreliability of component i, u_i, in the other type of consecutive-k-out-of-n system (say G system) for all i's (i.e., i=1,2,...n), given that both types of systems have the same k and n, then the reliability of one type of system is the same as the unreliability of the other type of system.

Proof of Theorem 6.1:

Define:

R_F: reliability of a consecutive-k-out-of-n:F system.

Q_F: $1-R_F$.

R_G: reliability of a consecutive-k-out-of-n:G system.

Q_G: $1-R_G$.

r_i: reliability of component i.

u_i: $1-r_i$.

X_i: state of component i.

$$X_i = \begin{cases} 0, & \text{component i fails with probability } u_i \\ 1, & \text{component i good with probability } r_i \end{cases}$$

$Q_F=1-R_F=Pr\{$at least k consecutive components fail$\}$

$R_G=1-Q_G=Pr\{$at least k consecutive components good$\}$

If the probability for X_i to get state 0 in the consecutive-k-out-of-n:F system is the same as the probability for X_i to get state 1 in the consecutive-k-out-of-n:G system for all i's (i.e., i=1, 2, ..., n), then it will hold that

$$Q_F = R_G$$

Q.E.D.

Based on Theorem 6.1, the procedure described below is given to compute the reliability of one type of consecutive-k-out-of-n system (say G system) in the general case.

<u>Procedure:</u>

1. Consider a consecutive-k-out-of-n:G system as a consecutive-k-out-of-n:F system and let r_i in the F system be equal to u_i in the G system for all i's.
2. Use the existing method for the F system to compute the system reliability R_F.
3. According to Theorem 6.1, $Q_G=R_F$, so the reliability of the consecutive-k-out-of-n:G system is $R_G=1-Q_G$.

It should be emphasized that there does not exist a relation in general cases between a consecutive-k-out-of-n:F system and a consecutive-k-out-of-n:G system. Knowing the reliability of one system does not lead to the reliability of the other system. Only when the condition holds that the component reliabilities in one type of system are equal to the corresponding component unreliabilities in the other type of system, is the reliability of one system equal to the reliability of the other system.

Theorem 6.1 also facilitates construction of bounds on reliability for one type of system (say G system) by using the methods of deriving bounds on reliability for the other type of system (say F system). A procedure for doing this is described below.

Define:

ℓ_F: lower bound on reliability of a consecutive-k-out-of-n:F system.

u_F: upper bound on reliability of a consecutive-k-out-of-n:F system.

ℓ_G: lower bound on reliability of a consecutive-k-out-of-n:G system.

u_G: upper bound on reliability of a consecutive-k-out-of-n:G system.

Procedure:

1. Consider a consecutive-k-out-of-n:G system as a consecutive-k-out-of-n:F system and replace r_i in the F system with u_i in the G system for all i's.
2. Use the existing methods for the F system to obtain bounds u_F and ℓ_F on reliability of the F system.
3. Apply Theorem 6.1, $Q_G = R_F$, to the bounds. Then we have:

$$\ell_G = 1 - u_F$$

$$u_G = 1 - \ell_F$$

B. Closed Formulas for Computing Reliability of a Circular Consecutive-k-out-of-n:G System with the i.i.d. Components

By using the procedure described in the previous section, the reliability of a circular consecutive-k-out-of-n:G system with equally reliable components can be obtained in the following way.

From the results for a circular consecutive-k-out-of-n:F system [13], the equations for a circular consecutive-k-out-of-n:G system follow:

$$R_C(n;k) = 0, \quad \text{if } n < k \qquad (6.1a)$$

$$R_C(k;k)=r^k \tag{6.1b}$$

$$R_C(n;k)=nur^k+r^n, \quad \text{if } k<n\le 2k \tag{6.1c}$$

$$R_C(n;k)=R_C(n-1;k)+ur^k[1-R_C(n-k-1;k)], \quad \text{if } n\ge 2k+1 \tag{6.1d}$$

Based on the above equations, if $n\le 2k$, the reliability of a circular consecutive-k-out-of-n:G system can be computed directly. In fact, if $2k+1\le n\le 3k$, the system reliability can also be obtained directly. It can be verified that

$$\begin{aligned}R_C(2k+1;k)&=R_C(2k;k)+ur^k[1-R_C(k;k)]\\&=2kur^k+r^{2k}+ur^k(1-r^k)\end{aligned}$$

$$\begin{aligned}R_C(n;k)&=R_C(n-1;k)+ur^k[1-R_C(n-k-1;k)]\\&=R_C(2k;k)+ur^k(1-r^k)\\&\quad+ur^k\{\sum_{i=1}^{n-2k-1}[1-(k+i)ur^k-r^{k+i}]\}\\&=ur^k(2k+1-r^k)+r^{2k}\\&\quad+ur^k\{\sum_{i=1}^{n-2k-1}[1-(k+i)ur^k-r^{k+i}]\}\end{aligned}$$

In general, we have

$$R_C(n;k)=0, \quad \text{if } n<k \tag{6.2a}$$

$$R_C(k;k)=r^k \tag{6.2b}$$

$$R_C(n;k)=nur^k+r^n, \quad \text{if } k<n\le 2k \tag{6.2c}$$

$$R_c(2k+1;k)=ur^k(2k+1-r^k)+r^{2k}$$

(6.2d)

$$R_c(n;k)=ur^k(2k+1-r^k)+r^{2k}$$

$$+ur^k\{\sum_{i=1}^{n-2k-1}[1-(k+i)ur^k-r^{k+i}]\}$$

if $2k+1 \leq n \leq 3k$

(6.2e)

$$R_c(n;k)=R_c(n-1;k)+ur^k[1-R_c(n-k-1;k)]$$

if $n \geq 2k+1$

(6.2f)

Therefore, if the condition of $n<3k+1$ holds in a circular consecutive-k-out-of-n:G system, the reliability of the system can be obtained without computing the reliabilities of the subsystems of n-1, n-2, ..., 2k+1 components.

C. Comparison of Reliabilities between the Linear Systems and the Circular Systems

In the consecutive-k-out-of-n:G systems with equally reliable components, the reliability of the circular system is always higher than that of the linear system for $n>k$. However, in the design process, it might be more difficult to design a circular system than a linear system. Consequently, it is desirable to know the difference in reliabilities between the two types of systems.

If $k<n\leq 2k$, for the linear system, we have

$R(n;k)=r^k+(n-k)ur^k$

and for the circular system, we have

$R_c(n;k)=r^n+nur^k$

Therefore, the difference of the two system reliabilities is

$R_c(n;k)-R(n;k)=kur^k-r^k+r^n$

It is clear that for a fixed value of k, as n increases, the difference between the two reliabilities becomes smaller. Table 6.1 shows the differences in reliabilities between the circular consecutive-k-out-of-n:G systems and the linear consecutive-k-out-of-n:G systems for different combinations of n and k values. All components in the systems have the same reliability r=0.8. From the table, we can see that for a given k, as the system size n becomes larger, the reliability of consecutive-k-out-of-n:G systems also increases, but the reliability difference between two types of systems becomes smaller. Figure 6.1 illustrates the fact.

This has an important implication in the system design process. When the system size is very large, it makes almost no difference whether to design a linear system or a circular system.

D. Properties of Consecutive-k-out-of-n Systems

Let us consider the consecutive-k-out-of-n systems in which all components have the same component reliability. In this situation, a

TABLE 6.1. Reliability difference between Rc(n;k) and R(n;k)

COMPONENT RELIABILITY = 0.800 Difference	k=2	k=3	k=4	k=5	k=6	k=7
n= 2	0.000000					
n= 3	0.128000	0.000000				
n= 4	0.025600	0.204800	0.000000			
n= 5	0.025600	0.122880	0.245760	0.000000		
n= 6	0.009216	0.057344	0.180224	0.262144	0.000000	
n= 7	0.005939	0.057344	0.127795	0.209715	0.262144	0.000000
n= 8	0.002662	0.036372	0.085852	0.167772	0.220201	0.251658
n= 9	0.001483	0.023789	0.085852	0.134218	0.186646	0.218104
n=10	0.000723	0.017917	0.065719	0.107374	0.159803	0.191260
n=11	0.000382	0.012045	0.050956	0.107374	0.138328	0.169785
n=12	0.000192	0.008321	0.040487	0.090194	0.121148	0.152606
n=13	0.000100	0.005885	0.033454	0.076450	0.121148	0.138862
n=14	0.000051	0.004050	0.026421	0.065455	0.107405	0.127867
n=15	0.000026	0.002817	0.021037	0.056659	0.095860	0.127867
n=16	0.000013	0.001965	0.016863	0.049622	0.086074	0.117311
n=17	0.000007	0.001362	0.013546	0.042585	0.077696	0.108163
n=18	0.000004	0.000947	0.010806	0.036674	0.070443	0.100141
n=19	0.000002	0.000659	0.008641	0.031664	0.064092	0.093020
n=20	0.000001	0.000458	0.006918	0.027375	0.057740	0.086619

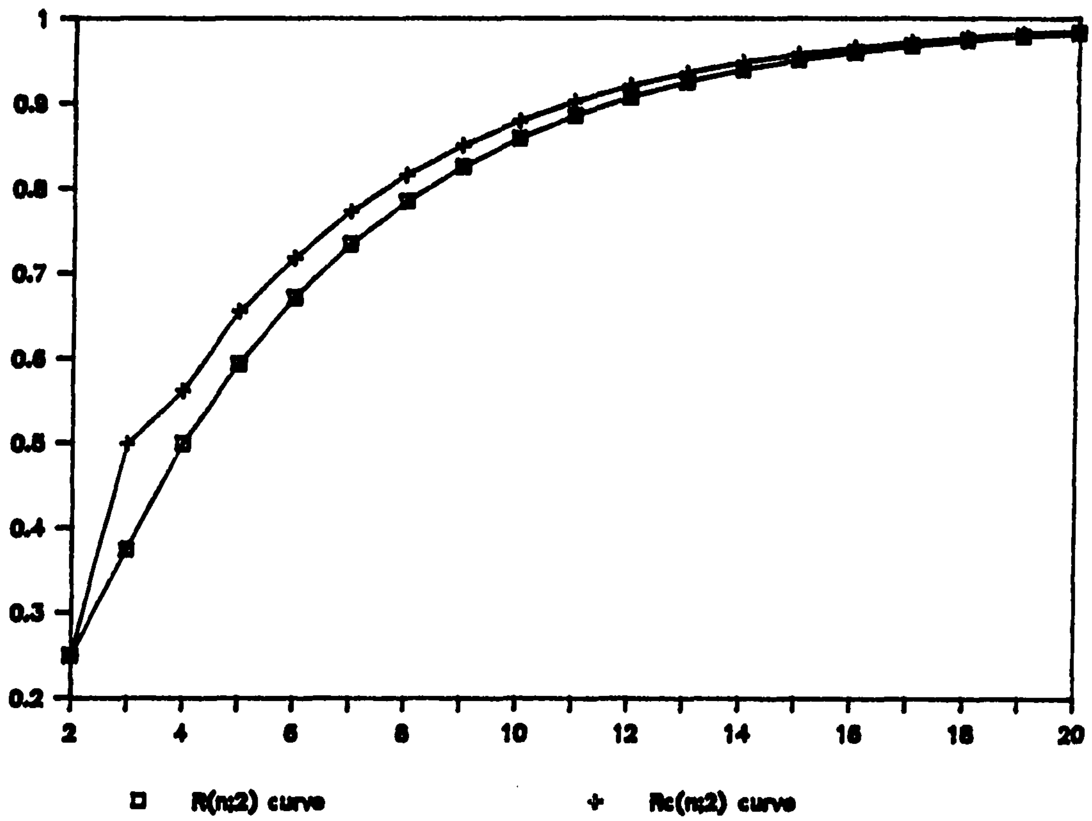

FIGURE 6.1. Reliability of the linear and circular consecutive-2-out-of-n:G systems as a fuction of n

consecutive-k-out-of-n system has 3 parameters, r, n and k. The last two parameters take only integer values.

For fixed values of n and k, as component reliability increases, the reliability of a consecutive-k-out-of-n system improves, regardless of whether it is a linear system, circular system, F system, or G system. The reliability of a linear consecutive-k-out-of-n:F system is always greater than that of the corresponding circular F system for n>k. In contrast, the reliability of a linear consecutive-k-out-of-n:G system is always less than that of the corresponding circular G system for n>k.

As system size n increases, the reliability of a consecutive-k-out-of-n:F system decreases; but the reliability of a consecutive-k-out-of-n:G system increases. As n increases, the reliability difference between the linear system and the circular system decreases, regardless of whether the system is a F system or G system.

As k increases, the reliability of a consecutive-k-out-of-n:F system becomes larger; but the reliability of a consecutive-k-out-of-n:G system becomes smaller.

VII. SYSTEM DESIGN

There are n components in a consecutive-k-out-of-n:G system. If all components in the system are interchangeable and not necessarily equally likely to fail, a problem of interest is to assign the components to the n positions in the system such that the system reliability is maximized. This consideration is very helpful in the pr :ess of system design.

A. Design of the Linear Consecutive-k-out-of-n:G Systems

There are n!/2 arrangements of n components in a linear consecutive-k-out-of-n:G system. The optimal configuration of a linear consecutive-k-out-of-n:G system is a design or an arrangement of n components such that the probability that the system functions is maximized.

Theorem 7.1:

The necessary condition for the optimal configuration of a linear consecutive-k-out-of-n:G system is as follows.

1. Arrange the components from position 1 to position min(k,n-k+1) in non-decreasing order of component reliability.
2. Arrange the components from position max(k,n-k+1) to position n in non-increasing order of component reliability.

Corollary 1:

If $n < 2k-1$ and a linear consecutive-k-out-of-n:G system has already been optimally designed, then the interchange of any two components in between component n-k and component k+1 inclusively does not affect the optimality of the system reliability.

A similar theorem, which is weaker than Theorem 7.1, states the necessary condition for the optimal configuration of a linear consecutive-k-out-of-n:G system if $n \leq 2k$, and is given as Theorem 1 in Appendix.

Proof of Theorem 7.1:

Let $m = \min(k, n-k+1)$

$j = i+1$

$1 \leq i < j \leq m$

$r_1 \leq r_2 \leq \ldots \leq r_i \leq r_j \leq \ldots \leq r_m$

By the pivotal decomposition theorem,

$$R(n;k) = r_i r_j R1 + r_i u_j R2 + u_i r_j R3 + u_i u_j R4$$

where, in G system,

$$R1 = R(r_1, \ldots, r_{i-1}, 1, 1, r_{j+1}, \ldots, r_n; k)$$

$$R2 = R(r_1, \ldots, r_{i-1}, 1, 0, r_{j+1}, \ldots, r_n; k)$$

$$= R(r_{j+1}, \ldots, r_n; k)$$

$$R3 = R(r_1, \ldots, r_{i-1}, 0, 1, r_{j+1}, \ldots, r_n; k)$$

$$= R(1, r_{j+1}, \ldots, r_n; k)$$

$$R4 = R(r_1, \ldots, r_{i-1}, 0, 0, r_{j+1}, \ldots, r_n; k)$$

$$= R(r_{j+1}, \ldots, r_n; k)$$

If we interchange components i and j, then

$R(n;k) = r_j r_i R1 + r_j u_i R2 + u_j r_i R3 + u_j u_i R4$

Therefore, the difference in system reliability before and after the interchange of components i and j is

$$(r_i u_j - r_j u_i)R2 + (u_i r_j - u_j r_i)R3$$

$$= (r_i - r_j)R2 + (r_j - r_i)R3$$

$$= (r_i - r_j)(R2 - R3)$$

Since $r_i \le r_j$ and $R2 < R3$, the difference is non-negative. Hence, the system reliability can not be improved by interchanging components i and j. If the system is optimally designed, interchange of components i and j for all i's and j's (i<j, not necessarily j=i+1) will not improve the system reliability.

Because of symmetry, the same argument holds for the rightmost min(k,n-k+1) components.

Q.E.D.

Proof of Corollary 1:

If n<2k-1, then n-k+1<k. From equation (2.1), we have

$$\begin{aligned} R(n;k) &= r_1 \ldots r_k + (1-r_1)r_2 \ldots r_{k+1} + \ldots \\ &\quad + (1-r_{n-k})r_{n-k+1} \ldots r_n \\ &= r_1 \ldots r_{n-k+1} \ldots r_k \\ &\quad + (1-r_1)r_2 \ldots r_{n-k+1} \ldots r_{k+1} \\ &\quad + \ldots \\ &\quad + (1-r_{n-k})r_{n-k+1} \ldots r_k \ldots r_n \end{aligned} \tag{7.1}$$

In equation (7.1), each term contains the product of the component reliabilities of component n-k+1 through component k inclusive. If we exchange any two components between component n-k and component k+1 (exclusive), the system reliability does not change. Therefore, if the system has already been optimally configured, the optimality of the system will not be affected by interchanging any two components between position n-k and position k+1.

Q.E.D.

Theorem 7.1 provides only the necessary condition for the optimal configuration of a linear consecutive-k-out-of-n:G system. It is not a sufficient condition as illustrated by the following example.

There are five components with different component reliabilities of 0.5, 0.6, 0.7, 0.8 and 0.9. The system works if and only if at least 3 consecutive components are good. There are, in total, 120 possible arrangements in this problem.

If we arrange the components from position 1 to position 5 as 0.5, 0.7, 0.9, 0.8, 0.6, which follows the rule given in Theorem 7.1, then the system reliability is 0.6966.

However, if the components are arranged as 0.5, 0.6, 0.9, 0.8, 0.7, which also meets the requirement in Theorem 7.1, then the system reliability is only 0.6876. Therefore, Theorem 7.1 provides only a necessary condition for the optimal system configuration.

Theorem 7.2:

In a linear consecutive-k-out-of-n:G system if $n \le 2k$, the optimal configuration of the system is

$$(1, 3, 5, \ldots, n, \ldots, 6, 4, 2) \tag{7.2}$$

given that $r_1 \le r_2 \le \ldots \le r_{n-1} \le r_n$.

According to the theorem, the least reliable component should be put in position 1, the second least reliable component in position n, the third least reliable one in position 2, the fourth least reliable one in position n-1, and so on. The most reliable component is in the middle position of the system. As long as it can be shown that the interchange of any two components in configuration (7.2) will not improve the system reliability, the configuration is the optimal one. Figure 7.1 is given to show that the approach is to have the interchanges of one component on the left side with any components on the right side. Then, do this for all other components on the left side. By symmetry, the approach is valid for all components on the right side.

Proof of Theorem 7.2:

Define $1 \le i < \min(k, n-k+1)$

$\max(k, n-k+1) < j \le n$

and $i = n-j+1$

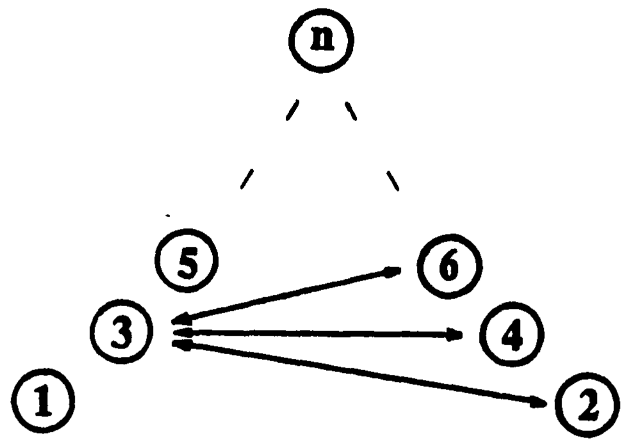

FIGURE 7.1. The optimal configuration

From configuration (7.2), we have

$$r_{j+1} \le r_i \le r_j \le r_{i+1} \le r_{j-1}$$

Configuration (7.2) satisfies the necessary condition given in Theorem 7.1. Furthermore, it follows

$$\min\{r_s\} \ge \max\{r_t\}$$

where $s \in \{\min(k,n-k+1), \ldots, \max(k,n-k+1)\}$

$t \in \{1, 2, \ldots, \min(k,n-k+1)-1\}$

or $t \in \{\max(k,n-k+1)+1, \ldots, n-1, n\}$

Since failure of any component between position min(k,n-k+1) and position max(k,n-k+1) will cause the system fail, the components with high reliability should be assigned to this range.

By the pivotal decomposition theorem,

$$R(n;k) = r_i r_j R1 + r_i u_j R2 + u_i r_j R3 + u_i u_j R4$$

where, in G system,

$R1 = R(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_{j-1},1,r_{j+1},\ldots,r_n;k)$

$R2 = R(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_{j-1},0,r_{j+1},\ldots,r_n;k)$

$= R(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_{j-1};k)$

$R3 = R(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_{j-1},1,r_{j+1},\ldots,r_n;k)$

$= R(r_{i+1},\ldots,r_{j-1},1,r_{j+1},\ldots,r_n;k)$

$R4 = R(r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_{j-1},0,r_{j+1},\ldots,r_n;k)$

$= 0$

If we interchange components i and j, then

$R(n;k) = r_j r_i R1 + r_j u_i R2 + u_j r_i R3 + u_j u_i R4$

The difference in system reliability before and after the interchange of components i and j is

$(r_i u_j - r_j u_i)R2 + (u_i r_j - u_j r_i)R3$

$= (r_i - r_j)R2 + (r_j - r_i)R3$

$= (r_i - r_j)(R2 - R3)$

Since $r_i \le r_j$ and $R2 < R3$, the difference can not be negative. Therefore, the system reliability can not be improved by interchanging components i and j.

If we interchange components i and j-1, it will violate the necessary condition for the optimal system design (Theorem 7.1), the configuration can not be optimal.

If we interchange components i and j+1, it still satisfies the necessary condition given in Theorem 7.1. The difference in system reliability before and after the interchange of components i and j+1 is

$(r_i - r_{j+1})(R'2 - R'3)$

where

$R'2 = R(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_j;k)$

$R'3 = R(r_{i+1},\ldots,r_j,1,r_{j+2},\ldots,r_n;k)$

Since $r_i \geq r_{j+1}$, if we can show that $R'2 \geq R'3$, the problem will be solved. In fact, this is done as follows.

$R(1,r_{i+1},\ldots,r_j;k) > R(r_{i+1},\ldots,r_j,1;k)$ (since $r_j < r_{i+1}$)

$R(r_{i-1},1,r_{i+1},\ldots,r_j;k) > R(r_{i+1},\ldots,r_j,1,r_{j+2};k)$

(since $r_{j+2} < r_{i-1}$)

...

Furthermore, the system related to R'2 has one more component than the system related to R'3. Hence, $R'2 > R'3$.

The same argument holds for different i's. Therefore, configuration (7.2) is the optimal one. In other words, it is a sufficient condition for the optimal design of a linear consecutive-k-out-of-n:G system with restriction of $n \leq 2k$.

Q.E.D.

B. Design of the Circular Consecutive-k-out-of-n:G Systems

In a circular consecutive-k-out-of-n:G system, each component precedes one component and succeeds another. In fact, there are $(n-1)!/2$ configurations for a circular system of n components. We are interested only in the configuration which maximizes the probability that the system works.

Theorem 7.3:

All arrangements of n components in a circular consecutive-k-out-of-(k+1):G system (equivalently, a circular consecutive-(n-1)-out-of-n:G system) give the same system reliability.

Proof of Theorem 7.3:

From equation (3.3c), we have

$$R_C(k+1;k) = \sum_{i=1}^{k+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{k+1} r_j \right) - k \prod_{i=1}^{k+1} r_i$$

$$= r_1 \ldots r_k + r_2 \ldots r_{k+1} + \ldots$$

$$+ r_{k+1} r_1 \ldots r_{k-1}$$

$$- k r_1 \ldots r_{k+1} \qquad (7.3)$$

Let us define $j=i+1$ and $1 \le i < j \le k+1$. If we interchange the positions of components i and j, the terms concerned in equation (7.3) before the interchange are

$$r_j r_{j+1} \ldots r_{k+1} r_1 \ldots r_{i-1}$$

$$+ r_{j+1} \ldots r_{k+1} r_1 \ldots r_{i-1} r_i$$

After the interchange, we have

$$r_i r_{j+1} \ldots r_{k+1} r_1 \ldots r_{i-1}$$

$$+ r_{j+1} \ldots r_{k+1} r_1 \ldots r_{i-1} r_j$$

The difference in system reliability before and after the interchange of components i and j is

$$(r_j - r_i)r_{j+1}\ldots r_{k+1}r_1\ldots r_{i-1}$$
$$+ r_{j+1}\ldots r_{k+1}r_1\ldots r_{i-1}(r_i - r_j)$$
$$= 0$$

The system reliability does not change by interchanging components i and j. The more interesting fact is that the system reliability does not change regardless of the relations between reliabilities of components i and j. Further, if components i and j are not neighbor components, each term in equation (7.3) contains the reliabilities of components i and j and therefore, the system reliability will not be affected by interchanging components i and j.

Q.E.D.

Theorem 7.3 examines that for a circular consecutive-k-out-of-(k+1):G system, the system reliability can not be improved by arranging the n components in the system, although there are k!/2 ways to arrange the components. This implies that any improvement in system reliability in such situations must come from some other effort instead of changing the system configuration.

Theorem 7.4:

The necessary condition for the optimal configuration of a circular consecutive-k-out-of-(k+2):G system is

$$(r_{(i)} - r_{(i+3)})\cdot(r_{(i+1)} - r_{(i+2)}) \geq 0, \quad \text{for } i = 1, 2, \ldots, k+2 \tag{7.4}$$

where $r_{(i)}$ represents the reliability of the component in position i, and $r_{(j)}=r_{(j-k-2)}$ if $j>k+2$.

Proof of Theorem 7.4:

The reliability of a circular consecutive-k-out-of-(k+2):G system can be calculated from equation (3.3d). Since this is the circular system where each component precedes one and succeeds another, we have the following relations.

$$r_0 = r_{k+2}$$

$$r_1 = r_{k+3}$$

From equation (3.3d), we have

$$R_c(k+2;k) = (1-r_{k+2})R(k+1;k) + r_{k+2}R_c(k+1;k)$$

$$+ \sum_{i=1}^{k} (1-r_i)(1-r_{i+1})\Big(\prod_{\substack{j=1\\ j\neq i\\ j\neq i+1}}^{k+2} r_j\Big)$$

$$= (1-r_{k+2})\Big[\prod_{i=1}^{k} r_i + (1-r_1)r_2\ldots r_{k+1}\Big]$$

$$+ r_{k+2}\Big[r_1\ldots r_k + \sum_{i=1}^{k} (1-r_i)\Big(\prod_{\substack{j=1\\ j\neq i}}^{k+1} r_j\Big)\Big]$$

$$+ \sum_{i=1}^{k} (1-r_i)(1-r_{i+1})\Big(\prod_{\substack{j=1\\ j\neq i\\ j\neq i+1}}^{k+2} r_j\Big)$$

$$= (1-r_{k+2}) \prod_{i=1}^{k} r_i + (1-r_1)r_2 \ldots r_{k+1}(1-r_{k+2})$$

$$+ r_{k+2}\left(\prod_{i=1}^{k} r_i\right) + \sum_{i=1}^{k} (1-r_i) \prod_{\substack{j=1 \\ j \neq i}}^{k+2} r_j$$

$$+ \sum_{i=1}^{k} (1-r_i)(1-r_{i+1})\left(\prod_{\substack{j=1 \\ j \neq i \\ j \neq i+1}}^{k+2} r_j\right)$$

$$= \prod_{i=1}^{k} r_i + \sum_{i=1}^{k} (1-r_i)\left(\prod_{\substack{j=1 \\ j \neq i}}^{k+2} r_j\right)$$

$$+ (1-r_1)r_2 \ldots r_{k+1}(1-r_{k+2})$$

$$+ \sum_{i=1}^{k} (1-r_i)(1-r_{i+1})\left(\prod_{\substack{j=1 \\ j \neq i \\ j \neq i+1}}^{k+2} r_j\right)$$

$$= \sum_{i=0}^{k+1}\left(\prod_{\substack{j=1 \\ j \neq i \\ j \neq i+1}}^{k+2} r_j\right) - \sum_{i=1}^{k+1}\left(\prod_{\substack{j=1 \\ j \neq i+1}}^{k+2} r_j\right) + \prod_{i=1}^{k+2} r_i$$

$$= r_1 \ldots r_k + r_2 \ldots r_{k+1} + \ldots + r_{k+2} r_1 \ldots r_{k-1}$$

$$- (r_1 \ldots r_{k+1} + \ldots + r_{k+2} r_1 \ldots r_k)$$

$$+ r_1 \ldots r_{k+2}$$

(7.5)

Assume that $j=i+1$ and $r_i \leq r_j$. If we interchange components i and j, the terms concerned before the interchange in equation (7.5) are

$$r_j r_{j+1} \ldots r_{j+k-1} + r_{i-k+1} \ldots r_{i-1} r_i$$

$$- r_j r_{j+1} \ldots r_{j+k} - r_{i-k} \ldots r_{i-1} r_i$$

After the interchange of components i and j, we have

$$r_i r_{j+1} \ldots r_{j+k-1} + r_{i-k+1} \ldots r_{i-1} r_j$$

$$- r_i r_{j+1} \ldots r_{j+k} - r_{i-k} \ldots r_{i-1} r_j$$

Then, the difference in system reliability before and after the interchange of components i and j is

$$(r_j - r_i r_{j+1} \ldots r_{j+k-1} + (r_i - r_j) r_{i-k+1} \ldots r_{i-1}$$

$$+ (r_i - r_j) r_{j+1} \ldots r_{j+k} + (r_j - r_i) r_{i-k} \ldots r_{i-1}$$

$$= (r_j - r_i)(1 - r_{j+k}) r_{j+1} \ldots r_{j+k-1} + (r_i - r_j)(1 - r_{i-k}) r_{i-k+1} \ldots r_{i-1}$$

$$= (r_j - r_i)(1 - r_{j+k}) r_{j+1} \ldots r_{j+k-1} - (r_j - r_i)(1 - r_{i-k}) r_{i-k+1} \ldots r_{i-1}$$

(Since $n=k+2$, then $j+k=i-1$, $j+k-1=i-2$, $i-k=j+1$ and $i-k+1=j+2$)

$$= (r_j - r_i)(1 - r_{i-1}) r_{j+1} \ldots r_{i-2}$$

$$= (r_j - r_i)(1 - r_{j+1}) r_{j+2} \ldots r_{i-1}$$

$$= (r_j - r_i)[(1 - r_{i-1}) r_{j+1} - (1 - r_{j+1}) r_{i-1}] r_{j+2} \ldots r_{i-2}$$

$$= (r_j - r_i)(r_{j+1} - r_{i-1}) r_{j+2} \ldots r_{i-2}$$

$= (r_j - r_i)(r_{j+1} - r_{i-1}) r_{i-k+1} \cdots r_{i-2}$

If the difference is non-negative, the system reliability can never be improved. Since $r_{i-1} \le r_i \le r_j \le r_{j+1}$, then the system reliability can not be improved by interchanging components i and j. Therefore, if the system is optimally designed, its reliability can not be improved by interchanging the positions of any two components in the system. As a result, the optimal system must satisfy

$$(r_{(i)} - r_{(i+3)})(r_{(i+1)} - r_{(i+2)}) \ge 0 \text{ for } i=1,2,\ldots,k+2$$

Q.E.D.

Theorem 7.4 provides a necessary condition for the optimal configuration of a circular consecutive-k-out-of-(k+2):G system and implies a way to improve the system reliability. The following procedure will be helpful in improving the reliability of such systems.

Step 1: Input ri for i=1,2,...,k+2.

Step 2: Set i=1.

Step 3: If $(r_{(i)} - r_{(i+3)})(r_{(i+1)} - r_{(i+2)}) \ge 0$,
go to step 5.

Step 4: Exchange components in positions i+1 and i+2;
set i=max(i-1,1);
go to step 3.

Step 5: If i=k+2, go to step 7.

Step 6: Set i=i+1;
go to step 3.

Step 7: Output the system configuration.

The above procedure is a heuristic approach and does not guarantee the optimal solution. This is illustrated by the following example.

Two configurations of a circular consecutive-4-out-of-6:G system are as follows:

(1, 2, 4, 6, 5, 3)

and

(1, 2, 5, 6, 4, 3)

If $r_1 \le r_2 \le \ldots \le r_6$, then both of them satisfy condition (7.4); but the first configuration gives a higher system reliability than the second one.

C. On Consecutive-k-out-of-n:F Systems

Theorem 7.5:

The necessary condition for the optimal configuration of a linear consecutive-k-out-of-n:F system are as follows.

1. Arrange the components from position 1 to position min(k,n-k+1) in non-decreasing order of component reliability.
2. Arrange the components from position max(k,n-k+1) to position n in non-increasing order of component reliability.

If $n < 2k-1$ and the linear consecutive-k-out-of-n:F system is the optimal configuration, then the interchange of any two components in between position n-k and position k+1 will not affect the optimality of the system reliability.

A similar theorem, but weaker than Theorem 7.4, is given as Theorem 2 in Appendix.

Proof of Theorem 7.5:

Let $m = \min(k, n-k+1)$

$j = i+1$

$1 \le i < j \le m$

$r_1 \le r_2 \le \ldots \le r_i \le r_j \le \ldots \le r_m$

By the pivotal decomposition theorem,

$R(n;k) = r_i r_j R1 + r_i u_j R2 + u_i r_j R3 + u_i u_j R4$

where

$R1 = R(r_1, \ldots, r_{i-1}, 1, 1, r_{j+1}, \ldots, r_n; k)$

$R2 = R(r_1, \ldots, r_{i-1}, 1, 0, r_{j+1}, \ldots, r_n; k)$

$R3 = R(r_1, \ldots, r_{i-1}, 0, 1, r_{j+1}, \ldots, r_n; k)$

$R4 = R(r_1, \ldots, r_{i-1}, 0, 0, r_{j+1}, \ldots, r_n; k)$

If we interchange components i and j, then

$R(n;k) = r_j r_i R1 + r_j u_i R2 + u_j r_i R3 + u_j u_i R4$

Therefore, the difference in system reliability before and after the interchange of components i and j is

$(r_i u_j - r_j u_i)R2 + (u_i r_j - u_j r_i)R3$

$= (r_i - r_j)R2 + (r_j - r_i)R3$

$= (r_i - r_j)(R2 - R3)$

$= (r_i - r_j)[(1-Q2) - (1-Q3)]$

$= (r_i - r_j)(Q3 - Q2)$

$= (r_i - r_j)[Q(r_1, \ldots, r_{i-1}, 0, 1, r_{j+1}, \ldots, r_n; k)$

$$- Q(r_1,\ldots,r_{i-1},1,0,r_{j+1},\ldots,r_n;k]$$

$$= (r_i-r_j)[Q(r_{j+1},\ldots,r_n;k) - Q(0,r_{j+1},\ldots,r_n;k)]$$

Since $r_i \le r_j$ and Q3 < Q2, the difference is non-negative. Hence, the system reliability can not be improved by interchanging components i and j. If the system is already optimally designed, interchange of components i and j for all i's and j's ($i<j$, not necessarily $j=i+1$) will not improve the system reliability.

Because of symmetry, the same argument holds for the rightmost min(k,n-k+1) components.

If $n<2k-1$, then $n-k+1<k$. Applying Theorem 6.1 to equation (2.1), we have

$$\begin{aligned} Q(n;k) = {} & u_1\ldots u_{n-k+1}\ldots u_k \\ & + (1-u_1)u_2\ldots u_{n-k+1}\ldots u_{k+1} \\ & + \ldots \\ & + (1-u_{n-k})u_{n-k+1}\ldots u_k\ldots u_n \end{aligned} \tag{7.6}$$

All terms in equation (7.6) contain the product of component reliabilities from position n-k+1 to position k. Therefore, interchanges of any two components in between position n-k and position k+1 will neither affect the unreliability of the system, nor the system reliability.

Q.E.D.

Theorem 7.6:

All configurations of a circular consecutive-k-out-of-(k+1):F system give the same system reliability.

Proof of Theorem 7.6:

Applying Theorem 6.1 to equation (3.3c), we have

$$Qc(k+1;k) = \sum_{i=1}^{k+1} \left(\prod_{\substack{j=1 \\ j\neq i}}^{k+1} u_j \right) - k \prod_{i=1}^{k+1} u_i$$

$$= u_1 \ldots u_k + u_2 \ldots u_{k+1} + \ldots$$

$$+ u_{k+1} u_1 \ldots u_{k-1} - u_1 \ldots u_{k+1}$$

(7.7)

Define $j=i+1$ and $1 \leq i < j \leq k+1$. If we interchange the positions of components i and j, the difference in system unreliability before and after the interchange is

$$(u_j - u_i) u_{j+1} \ldots u_{k+1} u_1 \ldots u_{i-1}$$

$$+ u_{j+1} \ldots u_{k+1} u_1 \ldots u_{i-1} (u_i - u_j) = 0$$

The system unreliability does not change regardless of the interchanges of two neighbor components. If components i and j are not neighbors, each term in equation (7.7) includes the unreliabilities of components i and j. Therefore, by interchanging components i and j, neither the system unreliability nor the system reliability changes.

Q.E.D.

Theorem 7.7:

The necessary condition for the optimal configuration of a circular consecutive-k-out-of-(k+2):F system is

$$(r_{(i)}-r_{(i+3)})(r_{(i+1)}-r_{(i+2)}) \leq 0 \quad \text{for } i=1,2,\ldots,k+2 \tag{7.8}$$

where $r_{(i)}$ represents the reliability of the component in position i and $r_{(j)}=r_{(j-k-2)}$, if $j>k+2$.

Proof of Theorem 7.7:

In a circular consecutive-k-out-of-(k+2):F system, the following relations hold:

$$u_0 = u_{k+2}$$

$$u_1 = u_{k+3}$$

Applying Theorem 6.1 to equation (7.5), we have

$$\begin{aligned} Q_c(k+2;k) = {} & u_1\ldots u_k + u_2\ldots u_{k+1} + \ldots \\ & + u_{k+2}\ldots u_{k-1} \\ & - (u_1\ldots u_{k+1} + \ldots + u_{k+2}u_1\ldots u_k) \\ & + u_1\ldots u_{k+2} \end{aligned} \tag{7.9}$$

Assume that $j=i+1$ and $r_i \leq r_j$ (or equivalently, $u_i \geq u_j$). If we interchange components i and j, then the difference in system unreliability before and after the interchange is

$$(u_j-u_i)(u_{j+1}-u_{i-1})u_{j+2}\ldots u_{i-2}$$

or equivalently,

$$(r_i-r_j)(r_{i-1}-r_{j+1})(1-r_{j+2})\ldots(1-r_{i-2})$$

Since $r_i \le r_j$, if $r_{i-1} \ge r_{j+1}$, then the difference is non-positive. This implies that the system unreliability can never be increased. Therefore, assuming that the system is optimally designed, the unreliability of the system can be increased by interchanging components i and i+1 for i=1, 2,...,k+2, i.e., it must be true that

$$(r_{(i)} - r_{(i+3)})(r_{(i+1)} - r_{(i+2)}) \le 0 \text{ for } i=1,2,\ldots,k+2$$

where $r_{(i)}$ represents the reliability of the component at position i.

Q.E.D.

VIII. A CASE STUDY

A railway station has 17 lines numbered from line 1 to line 17. The first 9 lines constitute the basic section which receives and sends trains, and the remaining 8 lines serve as the assembly section which organizes or reorganizes trains. The utilization density of a line can be considered as the probability that the line is not available.

All lines in the basic section have the same utilization density of u=0.35, and all lines in the assembly section have the utilization density of u=0.5.

The Master of the station has been informed that there will be a special train coming. However, because of over-limit loading of some vehicles, the neighbor lines of the line which receives the train must be empty, i.e., it is required that there are at least 3 consecutive lines empty so that the train can arrive. In addition, due to physical limitations, line 1, 9, 10 and 17 can not be used to receive the train. What is the probability that the special train can enter the station without delay, given that the assembly section can also accept the special train.

The problem, in fact, can be formulated as the reliability problem of a linear consecutive-k-out-of-n:G system. In other words, it is the problem of system availability. Taking into account the restrictions imposed on the station, we can regard the station as a linear consecutive-3-out-of-18:G system with a dummy line in between lines 9 and 10. Obviously, the dummy line assumes the utilization density of u=1.

In summary,

$u_1 = u_2 = \ldots = u_9 = 0.35$

$u_{10} = 1$

$u_{11} = \ldots = u_{18} = 0.5$

The utilization density of a line in this problem is similar to the unreliability of a component of a reliability system. Since $u_{10}=1$, the problem becomes one to find the condition probability that the system will work, i.e., to find

$$\Pr\{\text{accept the train / lines 9 and 10 can not be used}\}$$

$$= R(18;3/u_{10}=1)$$

Actually, we can find the probability, respectively, for a linear consecutive-3-out-of-9:G system (for lines 1 through 9) and for a linear consecutive-3-out-of-8:G system (for lines 10 through 17). Then, the probability of interest is obtained as follows:

$$R(18;3/u_{10}=1)=R(9;3)+R(8;3)-R(9;3)\cdot R(8;3)$$

$$= 0.744431 + 0.417969 - 0.744431 \times 0.417969$$

$$= 0.8512509$$

Using Theorem 5.1, the reliability importance of line in the basic section and the assembly section is calculated, respectively, and the results are given in Table 8.1.

From the results in Table 8.1, we know that lines 3, 7, 12 and 15 are the most important ones in the corresponding systems, respectively. If these lines are not available at the time the train comes, the probability of permitting the train to enter the station will be

TABLE 8.1. Reliability importance of lines in basic section and assembly section, respectively

basic section		assembly section	
line	importance	line	importance
1	0.07883745	10	0.10156250
2	0.17188859	11	0.21093750
3	0.27915323	12	0.33593750
4	0.20177740	13	0.25781250
5	0.21599090	14	0.25781250
6	0.20177740	15	0.33593750
7	0.27915323	16	0.21093750
8	0.17188859	17	0.10156250
9	0.07883745		

greatly reduced. For example, if line 3 is not empty, then lines 1 through 4 can not be used to receive the train. However, if line 1 is not empty, only line 2 can not be used. Reliability importance increases from line 1 to line 3 and decreases from line 7 to line 9 (last 3 lines) in the basic section. This fact confirms Corollary 1 of Theorem 5.1. It is the same for the assembly section. We see an 85% chance to accept the special train without delay.

If we assume that the dummy line can accept any trains with probability 0.00001, the probability of interest can be approximated as the following.

$u_1 = u_2 = \ldots = u_9 = 0.35$

$u_{10} = 0.99999$

$u_{11} = \ldots = u_{18} = 0.5$

Therefore,

$R(18;3) = 0.85125077$

Reliability importance for each line is given in Table 8.2.

The dummy line has the least reliability importance, and lines 3 and 7 are the most important. If we can reduce the utilization density of line 3 by 0.1, then the probability that the special train enters the station without delay will be increased to 0.86749828.

Applying equation (4.2) produces the lower bound and the upper bound on the probability of interest as shown below.

$r_1 = r_2 = \ldots = r_9 = 0.65$

$r_{10} = 0$

$r_{11} = \ldots = r_{18} = 0.5$

From equation (4.2), we have

$$\text{lower bound} = 1 - \prod_{j=0}^{5} \left(1 - \prod_{i=3j+1}^{3j+3} r_i\right)$$

$$= 1 - (1-r_1r_2r_3)(1-r_4r_5r_6)\ldots(1-r_{16}r_{17}r_{18})$$

$$= 1 - 0.2922158$$

$$= 0.707784$$

TABLE 8.2. Reliability importance of lines in station (whole system)

line	i	I_i
1	1	0.04588595
2	2	0.10004425
3	3	0.16247529
4	4	0.11743981
5	5	0.12571239
6	6	0.11743975
7	7	0.16247392
8	8	0.10004359
9	9	0.04588567
dummy	10	0.01788139
10	11	0.02595580
11	12	0.05390811
12	13	0.08585382
13	14	0.06588793
14	15	0.06588805
15	16	0.08585453
16	17	0.05390871
17	18	0.02595603

$$\text{upper bound} = 1 - \prod_{j=1}^{17} (1 - \prod_{i=0}^{2} r_{i+j})$$

$$= 1 - (1-r_1r_2r_3)(1-r_2r_3r_4)\ldots$$

$$(1-r_{15}r_{16}r_{17})(1-r_{16}r_{17}r_{18})$$

$$= 1 - 0.0474228$$

$$= 0.9525772$$

Therefore, the probability that the special train will enter the station without delay is no less than 0.707784, but no more than 0.9525772.

Now, let us consider the problem of optimal design for 9 lines in the basic section of the station. Since all lines have the same utilization density (equivalently, probability of not receiving trains), optimal design of the basic section does not occur.

On the average, about 3.15 trains stay in the basic section at any time. If we can control the utilization density for each line by changing schedules of train operation, optimal assignment of utilization density for all lines may take place. Suppose that we have 9 utilization densities to be assigned to 9 lines in the basic section.

$$u_i = 0.9 - i \times 0.05, \quad \text{for } i = 1, 2, \ldots, 9.$$

$$\prod_{i=1}^{9} u_i = 3.15$$

3.15 presents the fact that about 3.15 trains are standing in the basic section at any time. Equivalently, we have

$$r_i = 0.1 + i \times 0.05, \quad \text{for } i = 1, 2, \ldots, 9.$$

i.e.,

$r_1 < r_2 < \ldots < r_9$

where $r_i = 1 - u_i$ and means the probability that the line of interest is empty.

In this problem, we are interested in the assignment which maximizes the probability that the special train enters the basic section of the station without delay, i.e., we want the optimal configuration of a linear consecutive-3-out-of-9:G system.

Intuitively, less reliable components should be assigned to the end positions of a linear consecutive-k-out-of-n:G system, and more reliable components should go to the middle position of the system, since the middle positions produces more consecutive k-tuples of components than the end positions. Theorem 7.2 implies this fact for the case $n \leq 2k$.

For our problem of the linear consecutive-3-out-of-9:G system, 181440 possible configurations exist. Based on our best information, 12 configurations are listed in Table 8.3 with corresponding system reliabilities. Each configuration is selected in such a way that the 4 largest utilization densities are assigned to lines 1, 2, 8 and 9, and lines 3 and 7 are assigned by either the largest or the smallest ones among the remaining densities. The last 3 densities are assigned enumeratively to compare which assignment generates the best configuration. The configuration of # 8 in Table 8.3 gives the best system reliability among 12 configurations in this case.

Therefore, the assignment should be:

$r(1) = r_1$

TABLE 8.3. Configurations of the linear consecutive-3-out-of-9:G system

#	configuration	system reliability
1	(1,3,8,5,6,7,9,4,2)	0.81724846
2	(1,3,8,5,7,6,9,4,2)	0.81774491
3	(1,3,8,6,7,5,9,4,2)	0.81636608
4	(1,3,8,6,5,7,9,4,2)	0.81372321
5	(1,3,8,7,5,6,9,4,2)	0.81236666
6	(1,3,8,7,6,5,9,4,2)	0.81451291
7	(1,3,5,7,8,9,6,4,2)	0.82784170
* 8	(1,3,5,7,9,8,6,4,2)	0.82924467
9	(1,3,5,8,9,7,6,4,2)	0.82803583
10	(1,3,5,8,7,9,6,4,2)	0.82386810
11	(1,3,5,9,7,8,6,4,2)	0.82267875
12	(1,3,5,9,8,7,6,4,2)	0.82544333

$r(2) = r_3$

$r(3) = r_5$

$r(4) = r_7$

$r(5) = r_9$

$r(6) = r_8$

$r(7) = r_6$

$r(8) = r_4$

$r(9) = r_2$

where $r_{(i)}$ is the probability that line i is empty. This assignment maximizes the probability that the special train enters the basic section without delay.

In fact, the sufficient condition given by configuration (7.2) is not valid for the linear consecutive-k-out-of-n:G systems with $n > 2k$. Consider the following two configurations of a linear consecutive-2-out-of-8:G system.

$$(1, 3, 5, 7, 8, 6, 4, 2) \tag{8.1}$$

$$(1, 3, 6, 8, 7, 5, 4, 2) \tag{8.2}$$

If $(r_1, r_2, \ldots, r_8)$ = (0.45, 0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80), configuration (8.1) gives the system reliability of 0.943676 and configuration (8.2) 0.943672. The first configuration is slightly better than the second one. In contrast, if $(r_1, r_2, \ldots, r_8)$ = (0.111, 0.222, 0.333, 0.444, 0.556, 0.667, 0.778, 0.889), configuration

(8.1) gives the system reliability of 0.914592 and configuration (8.2) 0.915134. This time, the second configuration is better than the first one. In this case study, the best one in Table 8.3 conforms to configuration (7.2) by a coincidence.

In a linear consecutive-k-out-of-n system, we know that by the necessary conditions given in Theorems 7.1 and 7.5, the least reliable components should be assigned to the end positions. Therefore, at most (n-2)! configurations need to be examined. The effort to search for the optimal configuration is reduced by n(n-1)/2-1 times at least.

IX. SUMMARY AND DISCUSSION

In the past few years, there has been considerable interest in consecutive-k-out-of-n:F systems. A consecutive-k-out-of-n:F system is a sequence of n ordered components such that the system works if and only if less than k consecutive components fail.

This study introduces the concept of consecutive-k-out-of-n:G systems and develops the basic theory for consecutive-k-out-of-n:G systems reliability. A consecutive-k-out-of-n:G system consists of an ordered sequence of n components such that the system works whenever at least k consecutive components in the system are good. A consecutive-k-out-of-n:G system can be either a linear or a circular system, depending on whether all components are arranged on a line or on a circle.

Twelve theorems have been derived to establish and support the theory presented. Theorems 2.1 and 3.1 provide the methods to evaluate the reliability of linear and circular consecutive-k-out-of-n:G systems, respectively. The reliability importance of components measures the changing rate of system reliability with respect to a particular component in the system and therefore, indicates which component merits the most additional research and development to improve the overall system reliability with a minimum effort. Theorems 5.1 and 5.2 release the formulas to compute reliability importance of components in linear and circular consecutive-k-out-of-n:G systems, respectively.

Consecutive-k-out-of-n systems can be classified into two types, consecutive-k-out-of-n:F systems and consecutive-k-out-of-n:G systems. Theorem 6.1 implies a way to obtain solutions to one type of system by using the methods for the other type of system.

If all n components in a system are interchangeable, there exist many possible configurations, n!/2 for a linear system and (n-1)!/2 for a circular system. We are interested in the configuration which maximizes the probability that the system works. Theorems 7.1 and 7.5 supply the necessary conditions for the optimal configurations of linear consecutive-k-out-of-n:G systems and linear consecutive-k-out-of-n:F systems, respectively. In this case, the necessary conditions are the same for both G systems and F systems. Theorem 7.2 provides a sufficient condition for the optimal configuration of a linear consecutive-k-out-of-n:G system for $n \leq 2k$. Theorems 7.3 and 7.6 state that all configurations for both circular consecutive-k-out-of-(k+1):G systems and circular consecutive-k-out-of-(k+1):F systems give the same system reliability, respectively. Theorems 7.4 and 7.7 provide the necessary conditions for the optimal configurations of circular consecutive-k-out-of-(k+2):G systems and consecutive-k-out-of-(k+2):F systems ,respectively. The conditions are different in this case.

In general, reliability evaluations of consecutive-k-out-of-n:G systems are based on recursive approaches. If $n \leq 3k$ and all components in a system are equally reliable, then closed formulas are provided to evaluate system reliability directly for both linear and circular systems. Bounds on the reliability of consecutive-k-out-of-n systems are also studied since sometimes it is sufficient to compute the bounds on system reliability. An approximation to the reliability of a large linear consecutive-k-out-of-n:G system is proposed and a suggestion in this situation is given.

Although the results from this research turn out to be satisfactory, more investigation in this area is needed. For example, when all components in a system are not necessarily equally likely to fail, the reliability evaluation of circular consecutive-k-out-of-n:G system requires the reliability of linear subsystems. If a recursive approach that fully utilizes the reliability of circular subsystems can be found, it may provide some necessary conditions to design the circular systems better for more general situations.

X. REFERENCES

1. Antonopoulou, I. and S. G. Papastavridis. "Fast recursive algorithm to evaluate the reliability of a circular consecutive-k-out-of-n:F system". IEEE Trans. Reliability, R-36 (1987): 83-84.

2. Birnbaum, Z. W. "On the importance of different components in a multi-component system". In Multivariate Analysis-II. P. R. Krishnaiah, Editor. Academic Press, New York, 1969.

3. Bollinger, R. C. and A. A. Salvia. "Consecutive-k-out-of-n:F networks". IEEE Trans. Reliability, R-31 (1982): 53-56.

4. Chen, R. W. and F. K. Hwang. "Failure distributions of consecutive-k-out-of-n:F systems". IEEE Trans. Reliability, R-34 (1985): 338-341.

5. Chiang, D. T. and S. C. Niu. "Reliability of a consecutive-k-out-of-n:F system". IEEE Trans. Reliability, R-30 (1981): 87-89.

6. Derman, C., G. J. Lieberman and S. M. Ross. "On the consecutive-k-out-on-n:F system". IEEE Trans. Reliability, R-31, (1982): 57-63.

7. Feller, W. An Introduction to Probability Theory and its Applications. Vol. 1. 3rd ed. Wiley, New York, 1968.

8. Fu, J. C. "Reliability of a large consecutive-k-out-of-n:F system". IEEE Trans. Reliability, R-34 (1985): 127-130.

9. Fu, J. C. "Bounds for reliability of large consecutive-k-out-of-n:F systems with unequal component reliability". IEEE Trans. Reliability, R-35 (1986): 316-319.

10. Griffith, W. S. and Z. Govindarajulu. "Consecutive k-out-of-n failure systems: reliability, availability, component importance, and multistate extensions". American Journal of Mathematical and Management Sciences, 5, Nos. 1 & 2 (1985): 125-160.

11. Hwang, F. K. "Fast solutions for consecutive-k-out-of-n:F system". IEEE Trans. Reliability, R-31 (1982): 447-448.

12. Kontoleon, J. M. "Reliability determination of a r-successive-out-of-n system". IEEE Trans. Reliability, R-29 (1980): 437.

13. Lambiris, M. and S. G. Papastavridis. "Exact reliability formulas for linear & circular consecutive-k-out-of-n:F systems". IEEE Trans. Reliability, R-34 (1985): 124-126.

14. Malon, D. M. "Optimal consecutive-2-out-of-n:F component sequencing". IEEE Trans. Reliability, R-33 (1984): 414-418.

15. Malon, D. M. "Optimal consecutive-k-out-of-n:F component sequencing". IEEE Trans. Reliability, R-34 (1985): 46-49.

16. Papastavridis, S. G. "Upper and lower bounds for the reliability of a consecutive-k-out-of-n:F system". IEEE Trans. Reliability, R-35 (1986): 607-610.

17. Papastavridis, S. G. "The most important component in a consecutive-k-out-of-n:F system". IEEE Trans. Reliability, R-36 (1987): 266-268.

18. Salvia, A. A. "Simple inequalities for consecutive-k-out-of-n:F networks". IEEE Trans. Reliability, R-31 (1982): 450.

19. Shanthikumar, J. G. "Recursive algorithm to evaluate the reliability of a consecutive-k-out-of-n:F system". IEEE Trans. Reliability, R-31 (1982): 442-443.

20. Tillman, F. A., C. L. Hwang, and W. Kuo. Optimization of Systems Reliability. Marcel Dekker, New York, 1985.

21. Wei, V. K., F. K. Hwang and V. T. Sos. "Optimal sequencing of items in a consecutive-2-out-of-n system". IEEE Trans. Reliability, R-32 (1983): 30-33.

XI. ACKNOWLEDGEMENTS

I wish to express my thanks to Dr. Way Kuo for his guidance and assistance in this research.

This research is partially sponsored by the Iowa High Tech Research Program.

Thanks to each member of the committee: Dr. Mervyn Marasinghe, Dr. John Even, Dr. Howard Meeks and Dr. Thomas Barta for their support, and for teaching me many subjects during my study at Iowa State University.

Thanks also go to Mr. Daniel Kostka for his help in my writing of the thesis.

Thanks to my parents for their understanding.

Finally, I wish to thank my wife, Liping Chen, for her love, understanding and encouragement.

XII. APPENDIX

Theorem 1:

In a linear consecutive-k-out-of-n:G system, if $n \le 2k$, then the necessary condition for an optimal system configuration is:

1. Arrange component 1 through component min(k,n-k+1) in non-decreasing order of component reliability.
2. Arrange component max(k,n-k+1) through component n in non-increasing order of component reliability.

Proof of Theorem 1:

First, we consider the first min(k,n-k+1) components (from component 1 to component min(k,n-k+1)), and assume that those components have already been arranged in non-decreasing order of component reliability, i.e.,

$$r_1 \le r_2 \le \ldots \le r_{\min(k,n-k+1)}$$

Based on equation (2.1), the system reliability can be calculated as follows:

$$R(n;k) = \sum_{i=0}^{n-k} u_i \left(\prod_{j=i+1}^{i+k} r_i \right)$$

$$= r_1 \ldots r_k + (1-r_1) r_2 \ldots r_{k+1} + \ldots$$

$$+ (1-r_{n-k}) r_{n-k+1} \ldots r_n$$

(12.1)

where $u_0=1$.

Let us define i and j as follows:

$$j=i+1$$

$$1 \le i < j \le \min(k,n-k+1)$$

If we only interchange the positions of components i and j, the terms concerned in equation (12.1) before interchange are:

$$(1-r_i)r_j \ldots r_{k+i} + (1-r_j)r_{j+1} \ldots r_{k+j}$$

After interchange, we have

$$(1-r_j)r_i \ldots r_{k+i} + (1-r_i)r_{j+1} \ldots r_{k+j}$$

The difference in system reliability before and after the interchange of components i and j is

$$[(1-r_i)r_j - (1-r_j)r_i]r_{j+1} \ldots r_{k+i}$$
$$+ [(1-r_j) - (1-r_i)]r_{j+1} \ldots r_{k+j}$$
$$= (r_j-r_i)r_{j+1} \ldots r_{k+i}$$
$$+ (r_i-r_j)r_{j+1} \ldots r_{k+j}$$
$$= (r_j-r_i)r_{j+1} \ldots r_{k+i}(1-r_{k+j})$$

Since $r_j \ge r_i$, the difference due to the interchange of components i and j is non-negative. This implies that the interchange of components i and j will never improve the system reliability. Further, we can relax the assumption of $j=i+1$. As long as $i<j$, we can keep interchanging component i (or j) with its neighbor component until only components i and j have been interchanged. The choice of i or j depends on whether or not the system reliability is improved.

Now, let us consider the last min(k,n-k+1) components (from component max(k,n-k+1) to component n). Suppose that

$$r_{max(k,n-k+1)} \geq \ldots \geq r_{n-1} \geq r_n$$

From equation (2.2), we have

$$R(n;k) = \sum_{i=k}^{n} u_{i+1}\left(\prod_{j=i-k+1}^{i} r_i\right)$$

$$= r_{n-k+1}\ldots r_n + r_{n-k}\ldots r_{n-1}(1-r_n) + \ldots$$

$$+ r_1\ldots r_k(1-r_{k+1}) \tag{12.2}$$

where $u_{n+1}=1$.

Define

$j=i+1$

$\max(k,n-k+1) \leq i < j \leq n$

If we only interchange the positions of components i and j, the terms concerned in equation (12.2) before interchange are

$$r_{i-k+1}\ldots r_i(1-r_j) + r_{i-k}\ldots r_{i-1}(1-r_i)$$

After interchange of components i and j, we have

$$r_{i-k+1}\ldots r_{i-1}r_j(1-r_i)$$
$$+ r_{i-k}\ldots r_{i-1}(1-r_j)$$

Thus, the difference of system reliability before and after the interchange of components i and j is

$$[(r_i(1-r_j) - r_j(1-r_i)]r_{i-k+1}\ldots r_{i-1}$$
$$+ [(1-r_i) - (1-r_j)]r_{i-k}\ldots r_{i-1}$$

$= (r_i - r_j) r_{i-k+1} \cdots r_{i-1}$

$+ (r_j - r_i) r_{i-k} \cdots r_{i-1}$

$= (r_i - r_j) r_{i-k+1} \cdots r_{i-1} (1 - r_{i-k})$

Since $r_i \geq r_j$, then the system reliability can not be improved by interchanging components i and j. Also, as long as $i<j$ (not necessarily $j=i+1$) and $\max(k,n-k+1) \leq i < j \leq n$, the system reliability can not be improved by interchanging components i and j. Therefore, the components from position 1 to position min(k,n-k+1) should be arranged in non-decreasing order of component reliability, and the components form position max(k,n-k+1) to position n should be arranged in non-increasing order of component reliability.

If $n=2k$, components k and k+1 are in the middle positions of the system. Sometimes, the interchange of components k and k+1 will improve the system reliability, and the fact can be shown in the following.

From equation (2.1), we have

$$R(2k;k) = r_1 \cdots r_k + (1-r_1) r_2 \cdots r_{k+1} + \cdots + (1-r_k) r_{k+1} \cdots r_{2k} \tag{12.3}$$

If we exchange components k and k+1, the terms concerned in equation (12.3) before the interchange are

$$r_1 \cdots r_{k-1} r_k + (1-r_k) r_{k+1} r_{k+2} \cdots r_{2k}$$

After the interchange of components k and k+1, we have

$$r_1 \cdots r_{k-1} r_{k+1} + (1-r_{k+1}) r_k r_{k+2} \cdots r_{2k}$$

The difference in system reliability before and after the interchange is

$$r_1 \ldots r_{k-1}(r_k - r_{k+1})$$

$$+ r_{k+2} \ldots r_{2k}[(1-r_k)r_{k+1} - (1-r_{k+1})r_k]$$

$$= r_1 \ldots r_{k-1}(r_k - r_{k+1} + r_{k+2} \ldots r_{2k}(r_{k+1} - r_k)$$

$$= (r_{k+1} - r_k)(r_{k+2} \ldots r_{2k} - r_1 \ldots r_{k-1}) \qquad (12.4)$$

If the difference is non-negative, the system reliability can not be improved; otherwise the system reliability will be improved by interchanging components k and k+1.

Q.E.D.

Theorem 2:

If $n \le 2k$, the necessary condition for the optimal configuration of a linear consecutive-k-out-of-n:F system is:

1. Arrange the components from position 1 to position min(k,n-k+1) in non-decreasing order of component reliability.
2. Arrange the components from position max(k,n-k+1) to position n in non-increasing order of component reliability.

Proof of Theorem 2:

First, let us consider the components from position 1 to position min(k,n-k+1). Suppose that those components are arranged in non-decreasing order of component reliability, i.e.,

$r_1 \le r_2 \le \ldots \le r_{min(k,n-k+1)}$

or equivalently,

$u_1 \ge u_2 \ge \ldots \ge u_{min(k,n-k+1)}$

Applying Theorem 6.1 to equation (2.1), we have

$$Q(n;k) = \sum_{i=0}^{n-k} (1-u_i)\left(\prod_{j=i+1}^{i+k} u_j\right)$$

$$= u_1 \ldots u_k + (1-u_1)u_2 \ldots u_{k+1} + \ldots$$

$$+ (1-u_{n-k})u_{n-k+1} \ldots u_n$$

where $u_0=0$ for F system.

Define $j=i+1$ and $1 \le i < j \le min(k,n-k+1)$. If we only interchange the positions of components i and j, then the difference in system unreliability before and after the interchange is

$(u_j-u_i)u_{i+1} \ldots u_{k+1}(1-u_{k+j})$

Since $u_j \le u_i$, the difference is non-positive. As a result, the unreliability of the system is increased by interchanging components i and j. Therefore, components i and j should not be interchanged.

By the same token, the arrangement of components from position max(k,n-k+1) to position n in non-increasing order of component reliability will provide no chance of improving system reliability by interchanging any two components in this range. If the system has taken the optimal arrangement, the arrangement explained in Theorem 2 must be true.

Q.E.D.